KFUPM T161

Dept. Math & Stat Math 260

Quiz N°1

EX1: Suppose that a population $P(t)$ of birds is initially 100 and that it doubles after one year. We assume the rate of change of $P(t)$ is proportional to P. Find the size of $P(t)$ after 3 years.

Solution.

The DE governing this model is

$$\frac{dP(t)}{dt} = k\,P(t).$$

The solutions of this DE are given by:

$P(t) = c\,e^{kt}$, where $c \in \mathbb{R}$.

As $P(0) = 100$, we have $\boxed{c = 100}$.

Now $P(1) = 200$; so

$$200 = 100\,e^{k}$$

$\Rightarrow k = \ln(2)$.

Thus $P(t) = 100\,e^{t\ln(2)} = 100\,2^{t}$

Now, after 3 year the population of the birds will be

$$P(3) = 100(2^{3}) = \underline{800}.$$

EX2. If $A > 0$ and $y = 3\cos(Ax) + 2\sin(Ax)$

is a solution of the DE: $y'' + 16y = 0$,

Then find A.

Solution

$y' = -3A\sin(Ax) + 2A\cos(Ax)$

$y'' = -3A^2\cos(Ax) - 2A^2\sin(Ax)$

So $y'' + 16y = 0$ iff

$$-3A^2\cos(Ax) - 2A^2\sin(Ax) + 16\left[3\cos(Ax) + 2\sin(Ax)\right] = 0$$

$$\iff 3(16 - A^2)\cos(Ax) + 2(16 - A^2)\sin(Ax) = 0$$

This leads to $16 - A^2 = 0$, that is $A = \pm 4$.
As $A > 0$, we get $A = 4$.

Ex 3.
Solve the IVP: $\left(\dfrac{dy}{dx} = \dfrac{xy - 3x - 2y + 6}{xy - 2x - 3y + 6},\ y(0) = 1\right)$

Solution

As $xy - 3x - 2y + 6 = x(y-3) - 2(y-3)$
$= (x-2)(y-3)$

and

$xy - 2x - 3y + 6 = x(y-2) - 3(y-2)$
$= (x-3)(y-2)$

the DE is equivalent to

$\dfrac{dy}{dx} = \left(\dfrac{x-2}{x-3}\right)\left(\dfrac{y-3}{y-2}\right)$; and so it is separable

- Separating variables; we get

$$\left(\frac{x-2}{x-3}\right)dx = \left(\frac{y-2}{y-3}\right)dy$$

- Integrating, we have $\displaystyle\int\left(\frac{x-2}{x-3}\right)dx = \int\frac{y-2}{y-3}dy + c$

$$\Rightarrow \int\frac{(x-3)+1}{x-3}dx = \int\frac{(y-3)+1}{y-3}dy + c$$

$$\Rightarrow \int\left(1 + \frac{1}{x-3}\right)dx = \int\left(1 + \frac{1}{y-3}\right)dy + c$$

$$\Rightarrow x + \ln|x-3| = y + \ln|y-3| + c$$

The (I.C.) is $y(0) = 1$

This gives $\ln(3) = 1 + \ln(2) + c$, that is:

$$\boxed{c = -1 + \ln\left(\tfrac{3}{2}\right)}$$

Therefore, the solution is given implicitely by the equation:

$$x + \ln|x-3| = y + \ln|y-3| - 1 + \ln\left(\tfrac{3}{2}\right).$$

EX 4 A body of temperature 100°C is put in a room of temperature 35°C. If it takes the body 20 mn to cool down to 75°C, then after how much time the temperature of the body reaches 45°C?

Solution.

By Newton's Law of cooling (and warming), the DE governing this model is

$$\frac{dT(t)}{dt} = k(T(t) - A),$$ where $T(t)$ is the temperature of the body at time t and A is the temperature of the room ($A = 35$).

The solutions of the DE are given by:

$$T(t) = A + c\,e^{kt} = 35 + c\,e^{kt}$$

As $T(0) = 100$, we get $100 = 35 + c$; hence

$$\boxed{c = 65}$$

Now, as $T(20) = 75$, we have

$$75 = 35 + 65\,e^{20k}$$

$$\Rightarrow 40 = 65\,e^{20k} \Rightarrow k = \frac{1}{20}\ln\left(\frac{8}{13}\right)$$

Let τ be the time at which $T(\tau) = 45$, then

$$45 = 35 + 65\, e^{\frac{\tau}{20}\ln\left(\frac{8}{13}\right)}$$

$$\Rightarrow \quad 10 = 65\, e^{\frac{\tau}{20}\ln\left(\frac{8}{13}\right)}$$

$$\Rightarrow \quad \frac{\tau}{20}\ln\left(\frac{8}{13}\right) = \ln\left(\frac{2}{13}\right)$$

$$\Rightarrow \tau = 20\,\frac{\ln(13/2)}{\ln(13/8)}.$$

EX5: Solve the DE:

$$y'/4 + y = x^3 e^{-4x}$$

Solution. The given DE is linear: $y' + 4y = 4x^3 e^{-4x}$

- Finding an integrating Factor:

$$u(x) = e^{\int 4\,dx} = e^{4x} \text{ is an integ. fact.}$$

- Multiplying both sides of the normalized DE, we obtain

$$\left(e^{4x}y\right)' = 4x^3 e^{-4x} e^{4x} = 4x^3$$

By integrating, we have $e^{4x}y = x^4 + c$

So $y = x^4 e^{-4x} + c\, e^{-4x}$ is the general solution of the DE.