King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
(Math 260)

Code: 001

Final Exam
Term 161
Saturday, Jan 21, 2017
Building No. 57
Net Time Allowed: 180 minutes
07:00PM to 10:00PM

Name:		**ID:**	

(SHOW ALL YOUR STEPS AND WORK)

PART-I	**MCQ - PART**					
Q	**Answers**					**Points**
1	a	b	c	d	e	/5
2	a	b	c	d	e	/5
3	a	b	c	d	e	/5
4	a	b	c	d	e	/5
5	a	b	c	d	e	/5
6	a	b	c	d	e	/5
7	a	b	c	d	e	/5
8	a	b	c	d	e	/5
Total						**/40**

PART-II	**WRITTEN - PART**	
1		/11
2		/12
3		/13
4		/12
5		/12
6		/13
7		/13
8		/14
Total		**/100**
Grand Total		**/140**

MCQ PART

1) If $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 5 & 6 & 7 \\ 1 & 8 & 9 & 10 \\ 2 & 7 & 9 & 11 \end{bmatrix}$, then the sum of all the elements of the reduced row echelon form of A is:

A) 10
B) 19
C) 7
D) 22
E) 5

2) The space of all vectors (x, y, z, u, v) in $\mathcal{R}^5$ such that $4x + u = 3z + 2v$ has dimension

A) 4
B) 3
C) 5
D) 1
E) 2

3) Suppose that the rate of change of a population of a colony of bacteria is proportional to the population $P(t)$. At the start of an experiment, there are 6000 bacteria, and one hour later, the population has increased to 6400. Then the number of hours needed for the population to reach 10000 is:

A) $$\frac{ln\left(\frac{5}{4}\right)}{ln\left(\frac{8}{3}\right)}$$

B) $$\frac{ln\left(\frac{2}{5}\right)}{ln\left(\frac{64}{15}\right)}$$

C) $$\frac{ln\left(\frac{5}{2}\right)}{ln\left(\frac{5}{4}\right)}$$

D) $$\frac{\ln\left(\frac{5}{3}\right)}{\ln\left(\frac{16}{15}\right)}$$

E) $$\frac{ln\left(\frac{5}{4}\right)}{ln\left(\frac{64}{6}\right)}$$

4) For the system

$$x_1 - 4x_2 + x_3 - 4x_4 = 0$$
$$x_1 + 2x_2 + x_3 + 8x_4 = 0$$
$$x_1 + x_2 + x_3 + 6x_4 = 0,$$

then the solution space is:

A) $\{(0, s, 0, -s) + (0, -4t, -2t, 0, t) \mid s,t \in \mathcal{R}\}$

B) $\{(0, 0, -s, s) + (t, -2t, 0, 1) \mid s,t \in \mathcal{R}\}$

C) $\{(-s, 0, s, 0) + (-4t, -2t, 0, t) \mid s,t \in \mathcal{R}\}$

D) $\{(0, s, 0, 0) + (t, 0, t, t) \mid s,t \in \mathcal{R}\}$

E) $\{(s, -s, 0, 0) + (0, 0, -4t, 2t) \mid s,t \in \mathcal{R}\}$

5) If y is a solution of the Initial Value Problem

$4y'' - 8y' + 3y = 0,\ y(0) = 2,\ y'(0) = 2,$ then $y(\ln 4) =$

A) 10
B) 12
C) 4
D) 6
E) 8

6) Which of the following statements is true for any n x n matrices P, Q ?

A) $\det(P + Q) = \det P + \det Q$
B) $\det(P Q) = \det(Q P)$
C) $\det(n P) = 2^n \det P$
D) If $P \neq 0$ then P is invertible
E) If $P Q = 0$ and $P \neq 0$, then $Q = 0$

7) If $y(x)$ is the solution of the Initial value problem:
$\frac{dy}{dx} = (x + y - 1)^2$, $y(0) = 0$, then $y\left(\frac{\pi}{4}\right) =$

A) $1 - \frac{\pi}{4}$

B) $-1 - \frac{\pi}{4}$

C) $-1 + \frac{\pi}{4}$

D) $1 + \frac{\pi}{4}$

E) 1

8) A moving particle has acceleration $a(t) = \frac{1}{(1+t)^3}$, *initial position* $x_0 = 0$ and initial velocity $v_0 = 0$. The position function of the particle is $x(t) =$

A) $\frac{2}{t+1} + t - 2$

B) $\frac{1}{t+1} + t - 2$

C) $\frac{1}{t+1} + t - 1$

D) $\frac{2}{t+1} + 2(t-1)$

E) $\frac{t^2}{2(t+1)}$

WRITTEN PART

1) If y is a solution of the Initial Value Problem (11 Points)

$y''' - y'' + 9y' - 9y = 0,\ y(0) = 1,\ y'(0) = 9,\ y''(0) = -9$

Find $y\left(\frac{\pi}{12}\right)$.

Solution: characteristic equation of the DE is

$r^3 - r^2 + 9r - 9 = 0 \overset{(1)}{\Rightarrow} r^2(r-1) + 9(r-1) = 0 \Rightarrow (r-1)(r^2+9) = 0$

$\Rightarrow r = 1, \pm 3i$ (2)

The general solution is

$y(x) = c_1 e^x + c_2 \cos 3x + c_3 \sin 3x \longrightarrow (1)$ (1)

$\Rightarrow y'(x) = c_1 e^x - 3c_2 \sin 3x + 3c_3 \cos 3x \longrightarrow (2)$ (1)

$\Rightarrow y''(x) = c_1 e^x - 9c_2 \cos 3x - 9 \sin 3x \longrightarrow (3)$ (1)

Putting $y(0) = 1$ in (1), $y'(0) = 9$ in (2) and $y''(0) = -9$ in (3), we get

$c_1 + c_2 = 1 \longrightarrow (4)$, $c_1 + 3c_3 = 9 \longrightarrow (5)$ (2)

$-c_1 + 9c_2 = 9 \longrightarrow (6)$

Adding (4) and (6), we get $c_2 = 1$. Putting $c_2 = 1$ in (4), we get $c_1 = 0$ in (5), we get $c_3 = 3$. (1)

Therefore $y(x) = \cos 3x + 3 \sin 3x$ (1)

$\Rightarrow y\left(\frac{\pi}{12}\right) = \cos\frac{\pi}{4} + 3\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} = 2\sqrt{2}.$ (1)

2) Find the form of a particular solution of the following differential equation:

$$y^{(3)} - y^{(2)} + 16y' - 16y = \frac{1}{2}(\sin 4x)$$ (12 Points)

Solution: The characteristic equation of the associated homogenous equation is

$$r^3 - r^2 + 16r - 16 = 0 \Rightarrow (r-1)(r^2+16) = 0 \Rightarrow r = 1, \pm 4i$$

$\Rightarrow$ The general solution of homogenous differential equation is

$$y_c = C_1 e^x + C_2 \cos 4x + C_3 \sin 4x \longrightarrow (1)$$

One may seek a particular solution of the nonhomogenous DE of the form

$y_p = x^k(A\cos 4x + B\sin 4x)$ where k is the smallest +ve integer such that no term of y_p is duplicated in y_c.

Clearly $k=1$ and the general form of y_p is

$$y_p = x(A\cos 4x + B\sin 4x)$$

3) Let $A = \begin{bmatrix} r & \frac{1}{2} \\ 1-r & \frac{1}{2} \end{bmatrix}$, where r is a real number

(a) Find the eigenvalues (in terms of r)and associated eigenvectors of A. (8 Points)

Solution: $|A - \lambda I| = \begin{vmatrix} r-\lambda & \frac{1}{2} \\ 1-r & \frac{1}{2}-\lambda \end{vmatrix}$ 1 $= (\lambda - 1)(\lambda - (r - \frac{1}{2})) = 0$ 1

$\Rightarrow \lambda_1 = 1$ 1, $\lambda_2 = r - \frac{1}{2}$ 1

For $\lambda_1 = 1$, we solve $(A-I)V_1 = 0$ and get $V_1 = (\frac{1}{2}, 1-r)$ 2

For $\lambda_2 = r - \frac{1}{2}$, we solve $(A - (r-\frac{1}{2})I)V_2 = 0$ and we get

$V_2 = (-1, 1)$. 2

(b) Find the value of r for which A is not diagonalizable. (5 Points)

A has repeated eigenvalue $\lambda = 1$ if $r = \frac{3}{2}$. 2

In this case, we find that the dimension of the eigenspace is 1, 2 hence A is not diagonalizable iff $r = \frac{3}{2}$. 1

4) Use Cayley – Hamilton theorem to find

A^{-1} and A^3 for the matrix $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (12 Points)

Solution: $p(\lambda) = |A - \lambda I| = \begin{vmatrix} 1-\lambda & 3 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix}$ (1)

$$= (1-\lambda)(2-\lambda)^2 = (1-\lambda)(4+\lambda^2-4\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$ (3)

By Cayley – Hamilton theorem, we have

$$p(A) = -A^3 + 5A^2 - 8A + 4I = 0 \quad (2) \quad \Rightarrow A^3 = 5A^2 - 8A + 4I \quad (1)$$ (1)

Now $A^2 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3+6 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 9 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ (1)

Putting the values of A and A^2 in (1), we get

$$A^3 = 5 \begin{bmatrix} 1 & 9 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} - 8 \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 21 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

(1)

From (1), we have

$$4I = A^3 - 5A^2 + 8A \quad (1) \quad \Rightarrow A^{-1} = \frac{1}{4}\left[A^2 - 5A + 8I\right]$$ (1)

$$\Rightarrow A^{-1} = \frac{1}{4}\left[\begin{bmatrix} 1 & 9 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} - 5 \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}\right] = \frac{1}{2}\begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(1).

5

5) Solve the Initial Value Problem: $x' = 8x + 25y$, $y' = -x$; $x(0) = 3$, $y(0) = 0$.
(12 Points)

Solution: $y'' = -x' = -(8x+25y) = 8y' - 25y$ 1

$\Rightarrow y'' - 8y' + 25y = 0 \rightarrow (1)$. The characteristic equation

of the DE (1) is: $r^2 - 8r + 25 = 0$. 1

$\Rightarrow r = \dfrac{8 \pm \sqrt{64-100}}{2} = 4 \pm 3i$ 2

Therefore $y(t) = e^{4t}(c_1 \cos 3t + c_2 \sin 3t) \longrightarrow (2)$ 2

Putting $y(0)=0$ in (2), we get $c_1 = 0$. 1

$\Rightarrow y(t) = c_2 e^{4t} \sin 3t$

$\Rightarrow y'(t) = c_2 (3e^{4t} \cos 3t + 4e^{4t} \sin 3t)$

$\Rightarrow x(t) = -c_2(3e^{4t}\cos 3t + 4e^{4t}\sin 3t) \longrightarrow (3)$ 2

Putting $x(0) = 3$ in (3), we get $c_2 = -1$. 1

Hence the particular solution of the IVP is

$$x = 3e^{4t}\cos 3t + 4e^{4t}\sin 3t$$ 1

$$y = -e^{4t}\sin 3t.$$ 1

6) (a) If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, verify that the solutions (6 Points)

$$X_1 = e^{2t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ X_2 = e^{-t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},\ X_3 = e^{-t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

of the system X′ = AX are linearly independent.

Solution:

$$W(t) = e^{2t}\cdot e^{-t}\cdot e^{-t}\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{vmatrix} \quad (3)$$

$= 1(1) - 1(-1-1)$

$= 3 \neq 0 \Rightarrow X_1, X_2$ and X_3 are linearly independent. (2) (1)

(b) Solve the Initial Value Problem: X′ = AX, $X(0) = \begin{bmatrix} 10 \\ 12 \\ -1 \end{bmatrix}$ where

A is the matrix in Past (a). (7 Points)

Solution: The general solution of the system $X' = AX$ is

$$X(t) = c_1X_1(t) + c_2X_2(t) + c_3X_3(t) = c_1e^{2t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2e^{-t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3e^{-t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad (2)$$

$$\Rightarrow c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = X(0) = \begin{bmatrix} 10 \\ 12 \\ -1 \end{bmatrix} \Rightarrow \left.\begin{aligned} c_1 + c_2 &= 10 \\ c_1 + c_3 &= 12 \\ c_1 - c_2 - c_3 &= -1 \end{aligned}\right\} \to (1) \quad (2)$$

Solving the system (1), we get $c_1 = 7$, $c_2 = 3$, $c_3 = 5$. (1) (1) (1)

$$\Rightarrow X(t) = 7e^{2t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3e^{-t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 5e^{-t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{aligned} x_1(t) &= 7e^{2t} + 3e^{-t} \\ x_2(t) &= 7e^{2t} + 5e^{-t} \\ x_3(t) &= \_\_ + \_\_ + \_\_ \end{aligned}$$

7) Solve the Initial Value Problem: $X' = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} X, \quad X(0) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ (13 Points)

Solution: Let $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. Then

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4 = 1 + \lambda^2 - 2\lambda + 4 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 5 = 0 \Rightarrow \lambda = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i \quad (2)$$

Putting $\lambda = 1 - 2i$ in the eigenvector equation $(A - \lambda I)V = 0$, we get $\begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} = V$ (2)

$$\Rightarrow X(t) = Ve^{\lambda t} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(1-2i)t} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{t}(\cos 2t - i \sin 2t)$$

$$\Rightarrow X(t) = e^{t} \begin{bmatrix} \cos 2t - i \sin 2t \\ \sin 2t + i \cos 2t \end{bmatrix} \Rightarrow X_1(t) = e^{t} \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}, \quad (2)$$

$$X_2(t) = e^{t} \begin{bmatrix} -\sin 2t \\ \cos 2t \end{bmatrix} \quad (2)$$

$\Rightarrow$ The general solution is

$$X(t) = c_1 X_1(t) + c_2 X_2(t) \quad (1)$$

$$\Rightarrow x_1(t) = e^{t}(c_1 \cos 2t - c_2 \sin 2t) \longrightarrow (1) \quad (1)$$

$$x_2(t) = e^{t}(c_1 \sin 2t + c_2 \cos 2t) \longrightarrow (2) \quad (1)$$

Putting $x_1(0) = 0$ in (1) and $x_2(0) = 4$ in (2), we get $c_1 = 0$, $c_2 = 4$. (1) (1)

The particular solution of the DE is:

$$x_1(t) = -4e^{t} \sin 2t$$
$$x_2(t) = 4e^{t} \cos 2t$$

8) Find the general solution of the system $X' = AX$ where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix},$$ given that A has an eigenvalue $\lambda = 1$ of multiplicity 3.

(14 Points)

Solution: Eigenvector associated with $\lambda = 1$

$$A - I = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{R_1 + 2R_3} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix}.$$ (1)

Now $(A - I)\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \iff b = 0,\ a = -2c$. Therefore

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = r\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ (1). Thus $\dim(E_\lambda) = 1$ and $\lambda = 1$, is of defect 2. (1)

Let $v_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ (1). Let us find v_2 s.t. $(A - I)v_2 = v_1$. (1)

If $v_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then we get $a + 2c = 0,\ b = -1$.

choose $v_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ (2). Let us find v_3 s.t. $(A - I)v_3 = v_2$. (1)

If $v_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then we get $a + 2c = -1,\ b = 0$.

So we take $v_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$. (2)

Three linearly independent solutions are (1)

$X_1(t) = e^t v_1$ (1), $X_2(t) = e^t(t v_1 + v_2)$ (1), $X_3(t) = e^t\left(\frac{t^2}{2} v_1 + t v_2 + v_3\right)$

Thus the general solution of the DE is

$$X(t) = c e^t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c e^t \begin{bmatrix} -2t \\ -1 \\ t \end{bmatrix} + c e^t \begin{bmatrix} -t^2 - 1 \\ -t \\ t^2/2 \end{bmatrix}.$$ (1)

8) Find the general solution of the system $X' = AX$ where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix},$$ given that A has an eigenvalue $\lambda = 1$ of multiplicity 3.

(14 Points)

Solution: Eigenvector associated with $\lambda = 1$

$$A - I = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{R_1 + 2R_3} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix}.$$ (1)

Now $(A - I)\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \iff b = 0,\ a = -2c$. Therefore

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = r\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ (1). Thus $\dim(E_\lambda) = 1$ and $\lambda = 1$, is of defect 2. (1)

Let $v_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ (1). Let us find v_2 s.t. $(A - I)v_2 = v_1$. (1)

If $v_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then we get $a + 2c = 0,\ b = -1$.

Choose $v_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ (2). Let us find v_3 s.t. $(A - I)v_3 = v_2$. (1)

If $v_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then we get $a + 2c = -1,\ b = 0$.

So we take $v_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$. (2)

Three linearly independent solutions are (1)

$X_1(t) = e^t v_1$ (1), $X_2(t) = e^t(t v_1 + v_2)$ (1), $X_3(t) = e^t\left(\frac{t^2}{2} v_1 + t v_2 + v_3\right)$

Thus the general solution of the DE is

$$X(t) = c e^t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c e^t \begin{bmatrix} -2t \\ -1 \\ t \end{bmatrix} + c e^t \begin{bmatrix} -t^2-1 \\ -t \\ t^2/2 \end{bmatrix}.$$ (1)