King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
(Math 260)

Second Major Exam
Term 161
Thursday, December 01, 2016
Net Time Allowed: 100 minutes

Name:	Solution
ID:	
Section No:	
Serial No:	
Instructor's Name:	

(Show all your steps and work)

Question #	Marks
1	/11
2	/11
3	/13
4	/7
5	/12
6	/13
7	/10
8	/9
9	/14
Total	**/100**

(1) Write the homogenous system $\begin{matrix} x_1 - x_2 + 7x_4 + 3x_5 = 0 \\ x_3 - x_4 - 2x_5 = 0 \end{matrix}$ in the matrix form $AX = 0$.

Then find the solution in vector form $x = (x_1, x_2, x_3, x_4, x_5)$. [11 points]

Solution

- The matrix form of the system is $AX = 0$, where

$$A = \begin{bmatrix} 1 & -1 & 0 & 7 & 3 \\ 0 & 0 & 1 & -1 & -2 \end{bmatrix} \text{ and } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

4 2

- Here the matrix is in its reduced row echelon form;

so the solutions are given by:

$$x = (x_1, x_2, x_3, x_4, x_5) = (s - 7t - 3u, s, t + 2u, t, u),$$

1 1 1 1 1

where s, t, u are three free variables.

(2) Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$

i) Use elementary row operations to find the inverse A^{-1} *of* A. [7 points]

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right]$$

1

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

3

$$\xrightarrow[R_2 + 2R_3]{R_1 - 5R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 4 & -5 \\ 0 & 1 & 0 & -2 & -1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right].$$

1

So A is invertible and $A^{-1} = \begin{bmatrix} 6 & 4 & -5 \\ -2 & -1 & 2 \\ -1 & -1 & 1 \end{bmatrix}$. 2

ii) Use (i) to find a matrix X such that $AX = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$ [4 points]

As A is invertible, we get

$$X = A^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 & -5 \\ -2 & -1 & 2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$$

2

$$= \begin{bmatrix} 11 & -9 \\ -4 & 3 \\ -2 & 2 \end{bmatrix}$$

2

(3) (i) Given that $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 15,$ [8 points]

Evaluate the following determinant D:

$$D = \begin{vmatrix} \frac{c_1}{5} & a_1 + 3b_1 & 2b_1 \\ \frac{c_2}{5} & a_2 + 3b_2 & 2b_2 \\ \frac{c_3}{5} & a_3 + 3b_3 & 2b_3 \end{vmatrix}$$

$$D = \underbrace{\left(\frac{1}{5}\right) \times 2}_{2} \begin{vmatrix} c_1 & a_1 + 3b_1 & b_1 \\ c_2 & a_2 + 3b_2 & b_2 \\ c_3 & a_3 + 3b_3 & b_3 \end{vmatrix} \xlongequal{C_2 - 3C_3} \overbrace{\left(\frac{2}{5}\right) \begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix}}^{2}$$

$$\xlongequal[1]{swap(C_1, C_2)} -\left(\frac{2}{5}\right) \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} \xlongequal[1]{swap(C_3, C_2)} -\left(-\frac{2}{5}\right) \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\underbrace{= \left(\frac{2}{5}\right) \times 15 = 6.}_{2}$$

(ii) Let A and B be 4×4 matrices with $|A| = 3$ *and* $|B| = 4$. Find $|2A^T B^{-1}|$, where A^T is the transpose of A. [5 points]

$$|2A^T B^{-1}| = \underbrace{2^4}_{1} |A^T B^{-1}| = \underbrace{2^4 |A^T| \times |B^{-1}|}_{1}$$

$$= 2^4 \underbrace{|A|}_{1} \times \underbrace{\frac{1}{|B|}}_{1} = 16 \times 3 \times \frac{1}{4} = \underbrace{12.}_{1}$$

(4) Let $W = \left\{ (x,\ y) \in \mathbb{R}^2 \;\middle|\; \sin x = \cos\left(\frac{\pi}{2} + y\right) \right\}$

Is W a subspace of $\mathbb{R}^2$? Justify your answer. [7 points]

Solution.

2

Clearly $(\pi, \pi) \in W$ (as $\sin(\pi) = \cos\left(\frac{\pi}{2} + \pi\right) = 0$); but

3 $\left(\frac{1}{2}\right) \cdot (\pi, \pi) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right) \notin W$, as $\sin\left(\frac{\pi}{2}\right) = 1 \neq \cos\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = -1$

Thus W is not closed under scalar multiplication, consequently, W is not a subspace of $\mathbb{R}^2$.

2

(5) Express, if possible, the vector $t=(1,\ 2,\ -6)$ as a linear combination of the vectors $u=(0,\ 1,\ 2)$, $v=(1,\ 0,\ 2)$, $w=(1,\ 2,\ 0)$.: [12 points]

Solution.

Let us find $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$t=\alpha_1 u+\alpha_2 v+\alpha_3 w$; which is equivalent to the system

$$\begin{cases} \alpha_2+\alpha_3=1 \\ \alpha_1+2\alpha_3=2 \\ 2\alpha_1+2\alpha_2=-6 \end{cases}$$

4

The augmented matrix of the system is:

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 2 & 2 & 0 & -6 \end{array}\right] \xrightarrow[(1/2)R_3]{R_{1,2}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & -3 \end{array}\right]$$

$$\xrightarrow{R_3-R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -2 & -5 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -3 & -6 \end{array}\right]$$

$$\xrightarrow{(-1/3)R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[R_2-R_3]{R_1-2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

2 2 2

Thus $(\alpha_1, \alpha_2, \alpha_3)=(-2,-1,2)$; that is

$$\boxed{t=-2u-v+2w}$$ 2

(6) Suppose that v_1, v_2 *and* v_3 are linearly independent vectors. Determine whether

$u_1 = 2v_2 + 3v_3$, $u_2 = v_1 + v_3$, $u_3 = 2v_1 + v_2$ are linearly independent. Justify your answer.

[13 points]

Solution.

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0$; (3)

then $(\alpha_2 + 2\alpha_3)v_1 + (2\alpha_1 + \alpha_3)v_2 + (3\alpha_1 + \alpha_2)v_3 = 0$. (2)

As v_1, v_2, v_3 are linearly independent, we get

$$(*)\quad \begin{cases} \alpha_2 + 2\alpha_3 = 0 \\ 2\alpha_1 + \alpha_3 = 0 \\ 3\alpha_1 + \alpha_2 = 0 \end{cases}$$ (2)

The determinant of the matrix of coefficients is

$$D = \begin{vmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 3 & 1 & 0 \end{vmatrix} \xrightarrow{C_3 - 2C_1} \begin{vmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & -2 \end{vmatrix} = -\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} = 7 \neq 0$$

(4)

So the homogeneous system $(*)$ has only the trivial solution

$(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$.

Therefore, u_1, u_2, u_3 are linearly independent. (2)

(7) Find all possible values of k for which the vectors $(0, 1, 2, 1), (1, 0, 2, 3)$, $(2, 0, k, 1), (0, k, 1, 2)$ do not form a basis of $\mathbb{R}^4$. [10 points]

Solution.

Let us evaluate the determinant of the 4 vectors

(3)
$$D = \begin{vmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & k \\ 2 & 2 & k & 1 \\ 1 & 3 & 1 & 2 \end{vmatrix} \overset{C_3 - 2C_2}{=\!=\!=} \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & k \\ 2 & 2 & k-4 & 1 \\ 1 & 3 & -5 & 2 \end{vmatrix}$$

(3)
$$= - \begin{vmatrix} 1 & 0 & k \\ 2 & k-4 & 1 \\ 1 & -5 & 2 \end{vmatrix} \overset{C_3 - kC_1}{=\!=\!=} - \begin{vmatrix} 1 & 0 & 0 \\ 2 & k-4 & 1-2k \\ 1 & -5 & 2-k \end{vmatrix}$$

$$= - \begin{vmatrix} k-4 & 1-2k \\ -5 & 2-k \end{vmatrix} = - \left[(k-4)(2-k) + 5(1-2k)\right]$$

$$= - \left[-k^2 + 6k - 8 + 5 - 10k\right] = k^2 + 4k + 3. \text{ (1)}$$

The given vectors do not form a basis of $\mathbb{R}^4$ if and only if $D = 0$; that is $k^2 + 4k + 3 = 0$, equivalently $(k+1)(k+3) = 0$; this yields $\underline{k = -1}$ or $\underline{k = -3}$.

(1) (1)

(1) Therefore the given vectors do not form a basis of $\mathbb{R}^4$ if and only if $k = -1$ or $k = -3$.

(8) Find a second order differential equation of the form $ay'' + by' + cy = 0$ with constant coefficient such that $y = e^{2x}\left(c_1\, e^{x\sqrt{3}} + c_2\, e^{-x\sqrt{3}}\right)$ is the general solution of the differential equation. [9 points]

Solution.

The general solution of the DE is:

$$y = c_1\, e^{(2+\sqrt{3})x} + c_2\, e^{(2-\sqrt{3})x}.$$

So $r_1 = 2+\sqrt{3}$ (2), $r_2 = 2-\sqrt{3}$ (2) are the roots of the characteristic equation of the DE:

This charct. eqn is equivalent to:

$$\left(r - (2+\sqrt{3})\right)\left(r - (2-\sqrt{3})\right) = 0 \quad (2)$$

$$\Longleftrightarrow \boxed{r^2 - 4r + 1 = 0}$$

Thus the required DE is:

$$y'' - 4y' + y = 0. \quad (3)$$

(9) (i) Verify that the solutions $y_1 = e^x$, $y_2 = e^x \cos x$, $y_3 = e^x \sin x$ of the differential equation $y^{(3)} - 3y'' + 4y' - 2y = 0$ are linearly independent on $(-\infty, \infty)$.

Solution. $W(y_1, y_2, y_3)(x) =$ [6 points]

$$(3)\ \begin{vmatrix} e^x & e^x \cos x & e^x \sin x \\ e^x & e^x \cos x - e^x \sin x & e^x \sin x + e^x \cos x \\ e^x & -2e^x \sin x & 2e^x \cos x \end{vmatrix} = e^{3x} \begin{vmatrix} 1 & \cos x & \sin x \\ 1 & \cos x - \sin x & \sin x + \cos x \\ 1 & -2\sin x & 2\cos x \end{vmatrix}$$

$$\xrightarrow[R_3 - R_1]{R_2 - R_1} e^{3x} \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -2\sin x - \cos x & 2\cos x - \sin x \end{vmatrix}$$

$$= e^{3x} \begin{vmatrix} -\sin x & \cos x \\ -2\sin x - \cos x & 2\cos x - \sin x \end{vmatrix} = e^{3x}\left(-2(\sin x)\cos x + \sin^2 x + 2(\sin x)(\cos x) + \cos^2 x\right)$$

$$(2)\ = e^{3x} \neq 0$$

Hence y_1, y_2, y_3 are linearly independent. (1)

(ii) Use (i) to solve the initial value problem: [8 points]

$$y^{(3)} - 3y'' + 4y' - 2y = 0, \quad y(0) = 1,\ y'(0) = 0,\ y''(0) = 0.$$

Solution. The General solution of the given DE is

$$y = c_1 e^x + c_2 e^x \cos x + c_3 e^x \sin x$$

The Initial conditions are equivalent to the system

$$\begin{cases} c_1 + c_2 = 1 & (1) \quad 1 \\ c_1 + c_2 + c_3 = 0 & (2) \quad 1 \\ c_1 + 2c_3 = 0 & (3) \quad 1 \end{cases}$$

From (1), (2), we get $c_3 = -1$ (1); so from (3), we obtain $c_1 = -2c_3 = 2$, (2) and consequently $c_2 = -1$. (1)

Therefore, the unique solution of the given IVP is

$$y(x) = 2e^x - e^x \cos x - e^x \sin x. \quad (2)$$