King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
(Math 260)

First Major Exam
Term 161
Thursday, October 20, 2016
Net Time Allowed: 100 minutes

Name:	
ID:	
Section No:	
Serial No:	
Instructor's Name:	

(Show all your steps and work)

Question #	Marks
1	/11
2	/10
3	/13
4	/10
5	/12
6	/9
7	/12
8	/10
9	/13
Total	**/100**

(1) Use Gaussian Elimination to find all solutions of the system [11 points]

$$\begin{aligned} x+2y+3z &= 9 \\ 2x \qquad -2z &= -2 \\ 3x+2y+\ z &= 7 \end{aligned}.$$

(2) Use the method of elimination to determine whether the system [10 points]

$$\begin{aligned} 4x - 2y + 6z &= 0 \\ x - y - z &= 0 \\ 2x - y + 3z &= 0 \end{aligned}$$

is consistent or inconsistent. If it is consistent, find its solution.

(3) Find the reduced row echelon form of the matrix $A = \begin{pmatrix} 3 & 6 & 1 & 7 \\ 2 & 4 & 5 & 9 \\ 5 & 10 & 8 & 18 \end{pmatrix}$. [13 points]

(4) Verify that $y_p = (\cos x)\ln(\sec x + \tan x)$, *where* $0 < x < \frac{\pi}{2}$, is a solution of the differential equation $y'' + y = -\tan x$. [10 points]

(5) Solve the initial value problem [12 points]

$$\frac{dy}{dx} = \frac{xy - 5x - 2y + 10}{xy - 4x - 3y + 12}, \quad y(0) = 1.$$

(6) Find the position function $x(t)$ of a moving particle with acceleration $a(t) = 50\sin 5t$, initial position $x_0 = 8$ and initial velocity $v_0 = -10$. [9 points]

(7) Solve the initial value problem [12 points]

$$xy' = 2y + x^3 \cos x, \qquad y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}.$$

(8) Use an appropriate substitution to transform the differential equation [10 points]

$$\frac{y'}{y^{\frac{2}{5}}} + (\sin x)\, y^{\frac{3}{5}} = \cos x$$

into a linear first-order differential equation.

(9) Solve the differential equation $\left(6xy^3+2y^4\right)dx+\left(9x^2y^2+8xy^3+\cos y\right)dy=0$.

[13 points]

1) <u>Solution</u>. We change the augmented matrix of the system to a row-echelon form

$$\begin{bmatrix} 1 & 2 & 3 & 9 \\ 2 & 0 & -2 & -2 \\ 3 & 2 & 1 & 7 \end{bmatrix} \xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{bmatrix} 1 & 2 & 3 & 9 \\ 0 & -4 & -8 & -20 \\ 0 & -4 & -8 & -20 \end{bmatrix}$$

$$\xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 2 & 3 & 9 \\ 0 & -4 & -8 & -20 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{4}R_2} \begin{bmatrix} 1 & 2 & 3 & 9 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We take z as a free variable, $z=t$. Then $y=5-2t$ and $x=-1+t$.

The solutions are given by: $x=-1+t$, $y=5-2t$, $z=t$ (t an arbitrary parameter)

② **Solution.** This system is homogeneous, hence it is consistent, with at least the trivial solution $x=y=z=0$.

To find all its solutions we can use Gaussian or Gauss-Jordan elimination.

Let $A = \begin{bmatrix} 4 & -2 & 6 \\ 1 & -1 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ be its coefficient matrix. We have

$$A \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -1 & -1 \\ 4 & -2 & 6 \\ 2 & -1 & 3 \end{bmatrix} \xrightarrow{R_3 - \frac{1}{2}R_2} \begin{bmatrix} 1 & -1 & -1 \\ 4 & -2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 - 4R_1} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 10 \\ 0 & 0 & 0 \end{bmatrix}.$$

Take $z=t$, then $y=-5t$ and $x=-4t$. So the solutions are given by: $x=-4t$, $y=-5t$, $z=t$ (t arbitrary parameter)

③ Solution.

$$A \xrightarrow{R_1-R_2} \begin{bmatrix} 1 & 2 & -4 & -2 \\ 2 & 4 & 5 & 9 \\ 5 & 10 & 8 & 18 \end{bmatrix} \xrightarrow[R_3-5R_1]{R_2-2R_1} \begin{bmatrix} 1 & 2 & -4 & -2 \\ 0 & 0 & 13 & 13 \\ 0 & 0 & 28 & 28 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{28}R_2]{\frac{1}{13}R_2} \begin{bmatrix} 1 & 2 & -4 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 2 & -4 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1+4R_2} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$ this is the RREF of A.

(4) $$y_p' = (-\sin x)\ln(\sec x + \tan x) + (\cos x)\left(\frac{\sec x.\tan x + \sec^2 x}{\sec x + \tan x}\right)$$

$$= (-\sin x)\ln(\sec x + \tan x) + 1$$

$$y_p'' = (-\cos x)\ln(\sec x + \tan x) - \sin x\left(\frac{\sec x.\tan x + \sec^2 x}{\sec x + \tan x}\right)$$

$= -y_p - \sin x.\sec x$, hence $y_p'' + y_p = -\tan x$,

which means y_p is a solution of the DE

$$y'' + y = -\tan x.$$

(5) Solution. $\frac{dy}{dx} = \frac{(x-2)(y-5)}{(x-3)(y-4)}$, so the DE is

separable. We have: $\left(\frac{y-4}{y-5}\right)dy = \left(\frac{x-2}{x-3}\right)dx$

i.e. $\int\left(1+\frac{1}{y-5}\right)dy = \int\left(1+\frac{1}{x-3}\right)dx$

$$y + \ln|y-5| = x + \ln|x-3| + C$$

Initial condition gives: $C = 1 + \ln 4 - \ln 3$, hence

IVP solution is:

$$y + \ln|y-5| = x + \ln|x-3| + 1 + \ln\frac{4}{3}.$$

(6) <u>Solution</u>. The velocity $v(t) = \int a(t)\,dt = -10\cos 5t + C_1$

and the position is given by $x(t) = -\int 10\cos 5t\,dt + C_1 t$

i.e. $x(t) = -2\sin 5t + C_1 t + C_2$

We have $x(0) = 8$ and $v(0) = -10$, i.e. $C_2 = 8$ and $C_1 = 0$

Hence the position function is $x(t) = -2\sin 5t + 8$

(7) Solution. DE is $y' - \frac{2}{x} y = x^2 \cos x$, linear.

Integrating factor $\rho = e^{-\int \frac{2dx}{x}} = x^{-2}$

We obtain $\left(\frac{1}{x^2} y\right)' = x^{-2} \cdot x^2 \cos x = \cos x$.

Hence $\frac{y}{x^2} = \sin x + C$ (by integration).

$y(\pi/2) = \pi/2$ gives $C = \frac{2}{\pi} - 1$.

IVP solution is therefore: $y = x^2\left(\sin x + \frac{2}{\pi} - 1\right)$

(8) <u>Solution</u> DE is $y' + (\sin x)y = (\cos x)y^{2/5}$

This is a Bernoulli equation for which we use the substitution: $u = y^{3/5}$, $\frac{du}{dx} = \frac{3}{5} y^{-\frac{2}{5}} y'$

to obtain the linear DE:

$$\frac{du}{dx} + \frac{3\sin x}{5} u = \frac{3}{5} \cos x$$

(9) Let $M = 6xy^3 + 2y^4$, $N = 9x^2y^2 + 8xy^3 + \cos y$

then $M_y = 18xy^2 + 8y^3$, $N_x = 18xy^2 + 8y^3 = M_y$, hence

DE is exact.

There is a function F (of x and y) such that

$F_x = M$ and $F_y = N$. We have

$$F = \int M dx = 3x^2y^3 + 2xy^4 + g(y), \text{ some function } g.$$

From $F_y = N$ we obtain

$$9x^2y^2 + 8xy^3 + g'(y) = 9x^2y^2 + 8xy^3 + \cos y,$$

hence $g'(y) = \cos y$ and we can take $g(y) = \sin y$

(we do not need a constant of integration here).

Then DE solution is given by:

$$3x^2y^3 + 2xy^4 + \sin y = C$$