KFUPM | | T161
Dept. Math. & Stat | Math 232 | A.Y: 016/017

TEST N° 3

Exercise 1. Let $R = \{(x,y) \in \mathbb{R}^2 : |x| = y^2\}$

Is R a function from $\mathbb{R}$ to $\mathbb{R}$? why?

Solution.

For a given $x \in \mathbb{R}$, x is related to the elements $\pm\sqrt{|x|}$. So, for example 1 is related to 1 and -1.

It follows that R is not a function.

Exercise 2. Let $g : \mathbb{Q} \longrightarrow \mathbb{Q}$ be the function defined by

$$g(r) = 3r - 1, \quad \forall r \in \mathbb{Q}.$$

(i) Find $g^{-1}(\mathbb{N})$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$

(ii) Find $g^{-1}(E)$, where E is the set of all even integers.

Solution.

(i) $g^{-1}(\mathbb{N}) = \{r \in \mathbb{Q} \mid g(r) \in \mathbb{N}\} = \{r \in \mathbb{Q} : 3r - 1 = n, \text{ for some } n \in \mathbb{N}\}$

$$= \{r \in \mathbb{Q} \mid r = \frac{1+n}{3} \text{ for some } n \in \mathbb{N}\}$$

$$= \left\{\frac{1+n}{3} : n \in \mathbb{N}\right\}$$

(ii) $g^{-1}(E) = \{r \in \mathbb{Q} \mid g(r) \in E\} = \{r \in \mathbb{Q} \mid 3r - 1 = 2n, \text{ for some } n \in \mathbb{Z}\}$

$$= \left\{\frac{1+2n}{3} : n \in \mathbb{Z}\right\}$$

Exercise 3.

(1) Show that for all $a, b \in \mathbb{Z}$, we have

$$a \equiv b \pmod 7 \longleftrightarrow 5a+2 \equiv 5b+2 \pmod 7$$

(2) Deduce from (1), that $f : \mathbb{Z}_7 \longrightarrow \mathbb{Z}_7$, $\bar{a} \longmapsto \overline{5a+2}$

is a well defined injective function.

Is f a bijection? why?

Solution. (1)

- Assume $a \equiv b \pmod 7$; then $a - b = 7k$, for some $k \in \mathbb{Z}$.

So $(5a+2) - (5b+2) = 5(a-b) = 5(7k)$; hence

$7 \mid (5a+2) - (5b+2)$; that is to say $5a+2 \equiv 5b+2 \pmod 7$

- Conversely, assume $5a+2 \equiv 5b+2 \pmod 7$; then

$5a+2-(5b+2)$ is a multiple of 7.

Thus $5(a-b) = 7s$, for some $s \in \mathbb{Z}$.

As $7 \mid 5(a-b)$ and $\gcd(7,5) = 1$, we have (by Gauss Lemma) 7 divides $a-b$. Therefore $a \equiv b \pmod 7$

(2) • f is well defined?

If $\bar{x} = \bar{y}$, then $\overline{5x+2} = \overline{5y+2}$ (by (1)).

Hence f is well defined.

- f is injective.

Let $\bar{x}, \bar{y} \in \mathbb{Z}_7$. If $f(\bar{x}) = f(\bar{y})$ then $\overline{5x+2} = \overline{5y+2}$;

so by (1), $\bar{x} = \bar{y}$. Thus f is one-to-one.

- As f is injective $|f(\mathbb{Z}_7)| = |\mathbb{Z}_7| = 7$, but as $f(\mathbb{Z}_7)$ is a subset of $\mathbb{Z}_7$ with the same cardinality, we have $f(\mathbb{Z}_7) = \mathbb{Z}_7$.

Therefore f is onto.

We conclude that f is a bijection.

Exercise 4. Find the inverse of the bijection

$$f : \mathbb{R} \setminus \{4\} \longrightarrow \mathbb{R} \setminus \{3\}$$
$$x \longmapsto \frac{3x+1}{x-4}$$

Solution. Let $y \in \mathbb{R} \setminus \{3\}$. Let us solve the equation $f(x) = y$, for $x \in \mathbb{R} \setminus \{3\}$.

$$f(x) = y \Leftrightarrow \frac{3x+1}{x-} = y \Leftrightarrow xy - 4y = 3x + 1$$
$$\Leftrightarrow x(y-3) = 4y + 1$$
$$\Leftrightarrow x = \frac{4y+1}{y-3}$$

So f is invertible and $f^{-1} : \mathbb{R} \setminus \{3\} \longrightarrow \mathbb{R} \setminus \{4\}$

$$y \longmapsto f^{-1}(y) = \frac{4y+1}{y-3}$$

Exercise 5. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 3 & 6 & 5 & 4 & 1 & 2 \end{pmatrix}$

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 1 & 4 & 5 & 3 & 7 & 2 \end{pmatrix}$$

(i) Determine α^{-1}, β^{-1}, $\alpha^{-1}\beta^{-1}$, $\beta^{-1}\alpha^{-1}$

(ii) Determine $\alpha\beta$, $\beta\alpha$.

Solution.

(i) $\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 2 & 5 & 4 & 3 & 1 \end{pmatrix}$

$\beta^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 5 & 3 & 4 & 1 & 6 \end{pmatrix}$

$\alpha^{-1}\beta^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 2 & 5 & 4 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 5 & 3 & 4 & 1 & 6 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 1 & 4 & 2 & 5 & 6 & 3 \end{pmatrix}$

$\beta^{-1}\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 5 & 3 & 4 & 1 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 2 & 5 & 4 & 3 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 6 & 7 & 4 & 3 & 5 & 2 \end{pmatrix}$

(ii) $\alpha\beta = (\beta^{-1}\alpha^{-1})^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 7 & 5 & 4 & 6 & 2 & 3 \end{pmatrix}$

$(\beta\alpha) = (\alpha^{-1}\beta^{-1})^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 7 & 3 & 5 & 6 & 1 \end{pmatrix}$