KFUPM, DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 232: FINAL EXAM, TERM 161, JANUARY 14, 2017

07:00–10:00 pm

Name : ....

ID : ....

Exercise	**Points**
1	: **6**
2	: **8**
3	: **6**
4	: **8**
5	: **8**
6	: **6**
7	: **6**
8	: **8**
9	: **6**
10	: **6**
11	: **8**
12	: **6**
13	: **6**
14	: **6**
15	: **6**
Total	: **100**

Exercise 1 (6 pts)**.**

(1) Find the following intersection

$$I = \bigcap_{n=1}^{\infty} [-\frac{1}{n}, 2 + \frac{1}{n}).$$

(2) Find the following union

$$J = \bigcup_{n=2}^{\infty} [1 + \frac{1}{n}, n].$$

Exercise 2 (8 pts)**.**

(1) Using Mathematical Induction, show that 7 divides $3^{2n+2} - 2^{n+1}$, for each integer $n \geq 0$.

(2) Using techniques of congruences and the fact that $9 \equiv 2 \pmod 7$, show that 7 divides $3^{2n+2} - 2^{n+1}$, for each integer $n \geq 0$.

Exercise 3 (6 pts)**.** Let E be a set and X, Y be a partition of E.
Consider the function

$$\begin{array}{rccl} f: & \mathcal{P}(E) & \longrightarrow & \mathcal{P}(X) \times \mathcal{P}(Y) \\ & A & \longmapsto & (A \cap X, A \cap Y) \end{array}$$

Show that f is a bijection.

Exercise 4 (8 pts)**.** Find all integers x, y such that

$$17x + 11y = 5.$$

Exercise 5 (8 pts)**.** Show that the function $f : \mathbb{R} \setminus \{14\} \longrightarrow \mathbb{R} \setminus \{7\}$ defined by $f(x) = \dfrac{7x-1}{x-14}$ is a bijection and find its inverse function f^{-1}.

Exercise 6 (6 pts)**.** Find the number of integers between 1 and 500 that are neither multiple of 11 nor multiple of 13.

Exercise 7 (6 pts)**.** Write the prime factorization of 360, and then evaluate $\sigma(360)$ (the sum of all divisors of 360).

Is 360 a perfect number? why?

Exercise 8 (8 pts)**.**

(1) Show that if A is a finite set, then $\mid \mathcal{P}(A) \mid<\mid \mathbb{N} \mid$.
(2) Show that if A is an infinite set, then $\mid \mathbb{N} \mid\leq\mid A$.
(3) Show that there is no set A such that $\mathcal{P}(A)$ is denumerable.

Exercise 9 (6 pts)**.** Use Schröder-Bernstein Theorem to show the following facts: For every $x \in \mathbb{R}$ and every $q \in \mathbb{Q}$, we have

$$\mid \mathbb{R} \setminus \{x\} \mid = \mid \mathbb{R} \mid \text{ and } \mid \mathbb{Q} \setminus \{q\} \mid = \mid \mathbb{Q} \mid .$$

Exercise 10 (6 pts)**.** Let $(G, \star)$ be a group.

(1) Let a and b be two elements of G. Show that that if a and b commute, then so are a^{-1} and a^{-1}.

(2) Show that G is Abelian if and only if $(a \star b)^{-1} = a^{-1} \star b^{-1}$, for all $a, b \in G$.

Exercise 11 (8 pts)**.** Let $*$ be the binary operation defined on $\mathbb{Z}$ by :

$$a * b = a + b - 14.$$

(1) Show that $(\mathbb{Z}, *)$ is an Abelian group.
(2) Show that $(\mathbb{Z}, *)$ and $(\mathbb{Z}, +)$ are isomorphic.

Exercise 12 (6 pts)**.** Let $(G, \star)$ be a group and $a \in G$. Show that the

$$H = \{g \in G : a \star g = g \star a\}$$

is a subgroup of G.

Exercise 13 (6 pts)**.** Let H be a subgroup of a group G such that $\mid G \mid= 6$ and $\{e\} \subset H \subset G$. What are the possible values of $\mid H \mid$.

For each of these values give the number of the left-cosets modulo H.

Exercise 14 (6 pts)**.** Use the continuum hypothesis to show that $|\mathbb{R}\setminus\mathbb{Q}| = |\mathbb{R}|$.

Exercise 15 (6 pts)**.** Give an example of a finite nonabelian group and an example of an infinite nonabelian group.