1. **[12 points]** Verify that the set of functions $\{x, x^2, 1/x\}$ form a **fundamental set of solutions** of $x^3 y''' + x^2 y'' - 2xy' + 2y = 0$ on $(0, \infty)$. **Form the general solution**

First, Each function satisfies the DE: 2

$\boxed{y_1 = x}$ 2 $\Rightarrow y_1' = 1, \quad y_1'' = y_1''' = 0$

So $x^3(0) + x^2(0) - 2x(1) + 2x = -2x + 2x = \boxed{0}$

$\boxed{y_2 = x^2}$ 2 $\Rightarrow y_2' = 2x, \quad y_2'' = 2, \quad y_2''' = 0$

So $x^3(0) + x^2(2) - 2x(2x) + 2x^2 =$

2 $= 2x^2 - 4x^2 + 2x^2 = \boxed{0}$

$\boxed{y_3 = \frac{1}{x} = x^{-1}} \Rightarrow y_3' = -x^{-2}, \quad y_3'' = 2x^{-3}, \quad y''' = -6x^{-4}$

So $x^3(-6x^{-4}) + x^2(2x^{-3}) - 2x(-x^{-2}) + 2x^{-1} =$

$-6x^{-1} + 2x^{-1} + 2x^{-1} + 2x^{-1} = 0$

Second, Show $\{x, x^2, 1/x\}$ forms 2 a Linearly Ind. Set of functions

$W = \begin{vmatrix} x & x^2 & 1/x \\ 1 & 2x & -x^{-2} \\ 0 & 2 & 2x^{-3} \end{vmatrix} =$

Expand around C_1: 2

$x \begin{vmatrix} 2x & -x^{-2} \\ 2 & 2x^{-3} \end{vmatrix} - 1 \begin{vmatrix} x^2 & x^{-1} \\ 2 & 2x^{-3} \end{vmatrix} = x(4x^{-2} + 2x^{-2}) - (2x^{-1} - 2x^{-1})$

$= x(6x^{-2}) = \frac{6}{x} \neq \boxed{0}$

2. (a) [**5 points**] Verify that $y_{p_1} = 3e^{2x}$ and $y_{p_2} = x^2 + 3x$ are, respectively, particular solutions of

$$y'' - 6y' + 5y = -9e^{2x}$$

and

$$y'' - 6y' + 5y = 5x^2 + 3x - 16$$

a) We have $y_{p_1}' = 6e^{2x}$ and $y_{p_1}'' = 12e^{2x}$

So $y_{p_1}'' - 6y_{p_1}' + 5y_{p_1} = 12e^{2x} - 36e^{2x} + 15e^{2x} = -9e^{2x}$ 3

Also, $y_{p_2}' = 2X + 3$ and $y_{p_2}'' = 2 \Rightarrow$

$y_{p_2}'' - 6y_{p_2}' + 5y_{p_2} = 2 - 6(2X+3) + 5(X^2 + 3X)$
$= 5X^2 + 3X - 16$ 2

(b)[**5 points**] Use part(a) to find particular solutions of

$$y'' - 6y' + 5y = 5x^2 + 3x - 16 - 9e^{2x} \quad \text{—— I}$$

and

$$y'' - 6y' + 5y = -10x^2 - 6x + 32 + e^{2x} \quad \text{—— II}$$

1

By the Superposition Pri. for Nonhomo. eq. a particular Sol of I

$y'' - 6y' + 5y = 5X^2 + 3X - 16 - 9e^{2X}$ is $y_p = X^2 + 3X + 3e^{2X}$ 2

For II : $y_p = -2y_{p_2} - \frac{1}{9}y_{p_1} = -2X^2 - 6X + \frac{1}{3}e^{2X}$

(Combination of y_{p_1} and y_{p_2}) 2

3. **[8 points]** Given that $y_1 = x\,\sin(\ln x)$ is a solution to the DE: $x^2y'' - x'y + 2y = 0$. Use the **Reduction of Order Formula**, to find the second Linearly Independent Solution y_2.

D.E. Can be written as

$$y'' - \frac{1}{x}y + \frac{2}{x^2}y = 0 \quad (1)$$

So $P(x) = \frac{-1}{x}$

$$y_2 = +y_1(x)\int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx \quad (2)$$

$$= x\sin(\ln x)\int \frac{e^{+\int \frac{1}{x}dx}}{x^2\sin^2(\ln x)}\,dx$$

$$= x\sin(\ln x)\int \frac{x}{x^2\sin^2(\ln x)}\,dx \quad (2)$$

$$= x\sin(\ln x)\int \frac{\csc^2(\ln x)}{x}\,dx \quad (2)$$

[by substit. $u = \ln x$, $du = dx/x$]

+

[$\int \csc^2 x\,dx = -\cot x + C$]

$$= x\sin(\ln x)\left(-\cot(\ln x)\right)$$

$$= \boxed{-x\cos(\ln x)} \quad (1)$$

9

4. [redacted] points] (a) Given $y'' - 3y' + 2y = 5e^{3x}$.
If $y_1 = e^x$, is a solution to the associated homogeneous equation. Use the method of **Reduction of order** to find a second linearly independent solution $y_2(x)$ of the homogeneous equation.

Let $y = u(x)e^x$ 1 and substit. in D.E.

1 $y' = ue^x + u'e^x$ and $y'' = u''e^x + 2u'e^x + ue^x$ 1

So $y'' - 3y' + 2y = e^x u'' - e^x u' = 5e^{3x}$

Reduce: $u' = w$ 1 $\Longrightarrow w' - w = 5e^{2x}$: Linear first order

Solve: $e^{-\int -dx} = e^{-x} \Longrightarrow$ 1 $\frac{d}{dx}(e^{-x}w) = 5e^{x}$ 1

gives $e^{-x}w = 5e^{x} + c_1 \Longrightarrow$

$w = u' = 5e^{2x} + c_1 e^{x}$ 1 and $u = \frac{5}{2}e^{2x} + c_1 e^{x} + c_2$ 1

and $y_2 = ue^{x} = \boxed{\frac{5}{2}e^{3x} + c_1 e^{2x} + c_2 e^{x}}$ $\boxed{y_2 = e^{2x}}$ 1

3 [redacted] points] (b) Find a particular solution to the nonhomogeneous equation.

If You compare $y = y_c + y_p$

$\Longrightarrow \boxed{y_p = \frac{5}{2}e^{3x}}$ 3

5. **[6 points]** Find the general solution to the fourth order DE:

$$\frac{d^4 y}{dx^4} - 7\frac{d^2 y}{dx^2} - 18y = 0$$

Related Aux. Eq: $m^4 - 7m^2 - 18 = 0$ 1 (In Q. Form)

$(m^2-9)(m^2+2) = 0 \Rightarrow m = \pm 3, \pm\sqrt{2}\, i$ 1

So $y = c_1 e^{3x} + c_2 e^{-3x} + c_4 \cos(\sqrt{2}x) + c_4 \sin(\sqrt{2}x)$

1 1 1 1

6. **[6 points]** Find the general solution of $y''' + 6y'' + y' - 34y = 0$, if it is known that $y_1 = e^{-4x}\cos x$ is one solution.

Aux. Eq: $m^3 + 6m^2 + m - 34 = 0$ 1

$y_1 = e^{-4x}\cos x \Rightarrow$ One root is $-4+i \Rightarrow -4-i$ 1

another root ($y_2 = e^{-4x}\sin x$)

$(m-(-4+i))(m-(-4-i)) = m^2+8m+17$ (Q.P.) 1

$$\begin{array}{r|l} & m-2 \\ \hline m^2+8m+17 & m^3+6m^2+m-34 \\ & -m^3-8m^2-17m \\ \hline & -2m^2-16m-34 \\ & -2m^2-16m-34 = \boxed{0} \end{array}$$ 1

$\Rightarrow y_3 = e^{2x}$ since $(m-2)$ factor 1

Now $\boxed{y = c_1 y_1 + c_2 y_2 + c_3 y_3}$ 1

7. **[6 points]** Find the general solution of $2y''' + 7y'' + 4y' - 4y = 0$, if $m_1 = \frac{1}{2}$ is one root of its auxiliary equation.

Aux Eq: $2m^3 + 7m^2 + 4m - 4 = 0$, 1 $(m - \frac{1}{2})$ is factor 1

$2m^3 + 7m^2 + 4m - 4 = (m-\frac{1}{2})(m^2+4m+4)$ 2

$= (m - 1/2)(m+2)^2 \Rightarrow m_2 = m_3 = -2$

$\boxed{y = c_1 e^{\frac{1}{2}x} + c_2 e^{-2x} + c_3 x e^{-2x}}$ 2

8. **[6 points]** Find a differential operator that annihilates
$13x + 9x^2 - \sin 4x + e^{-x} + 2x\,e^x - x^2\,e^x$

$13x + 9x^2 \longrightarrow D^3$ —— 1

$\sin 4x \longrightarrow (D^2 + 16)$ —— 1

$e^{-x} \longrightarrow (D+1)$ —— 1

$xe^x \longrightarrow (D-1)^2$

$x^2e^x \longrightarrow (D-1)^3$

$(D-1)^3$ Ann. — 1
Both fns 1

$f(D) = D^3(D^2+16)(D+1)(D-1)^3$
is the Annihilator 1

9. **[12 points]** Solve the IVP

$$y'' - 5y' = x - 2 \quad \text{subject to} \quad y(0) = 0,\ y'(0) = 2$$

Aux. Eq: $m^2 - 5m = 0$ 1 $\Longrightarrow m = 0, 5 \Longrightarrow y_c = c_1 + c_2 e^{5x}$ 2

D^2 Annihilates $(x - 2)$ $[D^2 = 0 \Longrightarrow 0, 0]$ 1

$y_p = Ax + Bx^2$ (repeated zero) 1

Substit. y_p into DE

$y_p' = A + 2Bx$, $y_p'' = 2B$ 1

$2B - 5(A + 2Bx) = x - 2$

$2B - 5A - 10Bx = x - 2 \Longrightarrow \begin{cases} -10B = 1 \\ B = -1/10 \end{cases}$ 1

$\begin{cases} 2B - 5A = -2 \\ \text{so } A = 9/25 \end{cases}$ 1

So $\boxed{y = c_1 + c_2 e^{5x} + \frac{9}{25}x - \frac{1}{10}x^2}$

To Solve the IVP $y(0) = 0 \Longrightarrow \boxed{c_1 + c_2 = 0}$ 1

$y'(0) = 2$, $y' = 5c_2 e^{5x} + \frac{9}{25} - \frac{2}{10}x$

$2 = 5c_2 + \frac{9}{25} \Longrightarrow \boxed{c_2 = 41/125}$ 1

$$\boxed{y = \frac{-41}{125} + \frac{41}{125}e^{5x} + \frac{9}{25}x - \frac{1}{10}x^2}$$ 2

10. **[10 points]** Solve the DE by **Variation of Parameters**: $y'' + y = \cos^2 x$

Aux. Eq.: $m^2 + 1 = 0 \Rightarrow m = \pm i$ 1

So $y_c = c_1 \cos x + c_2 \sin x$, $W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$y_1 = \cos x$, $y_2 = \sin x$

$f(x) = \cos^2 x$, 1

1 $u_1' = \frac{W_1}{W} = \frac{-y_2 f(x)}{W} = -\sin x \cos^2 x$

1 $u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W} = \cos x . \cos^2 x = \cos^3 x$

$u_1 = \int \sin x \cos^2 x \, dx$ (by substit.)

2 $= -\int \sin x \cos^2 x \, dx = \frac{1}{3}\cos^3 x$ $[u = \cos x]$

$u_2' = \cos^3 x = \cos x(1 - \sin^2 x) = \cos x - \cos x \sin^2 x$

Same method $\Rightarrow u_2 = \sin x - \frac{1}{3}\sin^3 x$ 2

Now, $y = \underbrace{c_1 \cos x + c_2 \sin x}_{y_c} + \underbrace{u_1 y_1 + u_2 y_2}_{y_p}$,

$$y = \underbrace{c_1 \cos x + c_2 \sin x}_{y_c} + \underbrace{\frac{1}{3}\cos^4 x + \sin^2 x - \frac{1}{3}\sin^4 x}_{y_p}$$

1 1

2

11. **[6 points]** Solve the DE:

$$x^3 y''' - 6y = 0$$

3^{rd} order, Cauchy, assume $y = x^m \Rightarrow$ 1

$y' = m x^{m-1}$, $y'' = m(m-1)x^{m-2}$, $y''' = m(m-1)(m-2)x^{m-3}$

substit. in D.E. 1

$m(m-1)(m-2)x^{m-3} \cdot x^3 - 6x^m = 0$

2 $\Rightarrow m(m-1)(m-2) - 6 = m^3 - 3m^2 + 2m - 6$

$= (m-3)(m^2+2) = 0$, $m = 3, \pm\sqrt{2}i$

Thus:

2 $y = c_1 x^3 + c_2 \cos(\sqrt{2}\ln x) + c_3 \sin(\sqrt{2}\ln x)$

12. **[6 points]** Given the Cauchy-Euler DE:

$$x^2y'' - 4xy' + 6y = \ln x^2$$

Use the substitution $x = e^t$, to transform it to a differential equation with constant coefficients. 1

$x = e^t \Leftrightarrow t = \ln x$, by C.R.: $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$

also, $\frac{d^2y}{dx^2} = $ 1 $\frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$ (C.R. + Pro. Rule)

sub. into D.E. 2 $\boxed{\ln e^t = t}$

$x^2\left(\frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right) - 4x\left(\frac{1}{x}\frac{dy}{dt}\right) + 6y = 2t$

$\boxed{\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 2t}$ 2