Math201.05, Quiz #3 & 4, Term 161

Name: Solutions

ID #: **Serial #:**

1. [6 points] Find the local maximum and minimum values and saddle points of $f(x,y) = x^2 + y^2 + x^2y + 2$.

- $f_x(x,y) = 2x + 2xy \quad , \quad f_y(x,y) = 2y + x^2$
- f_x & f_y exist at all points (x,y) in the xy-plane.
- $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 2x + 2xy = 0 & \text{— (1)} \\ 2y + x^2 = 0 & \text{— (2)} \end{cases}$ 1

  - $(1) \Rightarrow 2x(1+y) = 0 \Rightarrow x = 0$ or $y = -1$ 2
  - $x = 0 \xRightarrow{(2)} 2y + 0 = 0 \Rightarrow y = 0 \Rightarrow \boxed{(0,0)}$
  - $y = -1 \xRightarrow{(2)} -2 + x^2 = 0 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2} \Rightarrow \boxed{(-\sqrt{2}, -1), (\sqrt{2}, -1)}$

- $f_{xx}(x,y) = 2 + 2y, \; f_{yy}(x,y) = 2 \; , \; f_{xy}(x,y) = 2x$

  $D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - [f_{xy}(x,y)]^2$ 1

  $= 4 + 4y - 4x^2$

- $D(0,0) = 4 > 0$ & $f_{xx}(0,0) = 2 > 0 \Rightarrow f$ has a local min. at $(0,0)$. 1

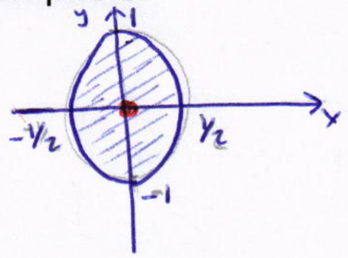
  the local min. value is $f(0,0) = 2$

- $D(-\sqrt{2}, -1) = 4 - 4 - 4(2) = -8 < 0 \Rightarrow f$ has a saddle point at $(-\sqrt{2}, -1)$ 1
- $D(\sqrt{2}, -1) = 4 - 4 - 4(2) = -8 < 0 \Rightarrow$ ———— at $(\sqrt{2}, -1)$.

  , $f(-\sqrt{2}, -1) = 2 + 1 - 2 + 2 = 3$

  $f(\sqrt{2}, -1) = 2 + 1 - 2 + 2 = 3$

2. [6 points] Find the extreme values of $f(x,y) = e^{-xy}$ on the region $\{(x,y): x^2 + 4y^2 \le 1\}$. **Hint:** you may use Lagrange Multipliers.



- Critical points (inside the ellipse)
  - $f_x(x,y) = -y e^{-xy}$, $f_y(x,y) = -x e^{-xy}$
  - f_x & f_y exist at all pts (x,y) in the xy-plane
  - $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} -y e^{-xy} = 0 \\ -x e^{-xy} = 0 \end{cases} \xrightarrow{e^{-xy} \neq 0} \begin{cases} y = 0 \\ x = 0 \end{cases}$

    $\Rightarrow \boxed{(x,y) = (0,0)}$ inside the ellipse

2

- The boundary $x^2 + 4y^2 = 1$

We use Lagrange Multipliers with

$$f(x,y) = e^{-xy} \quad , \quad g(x,y) = x^2 + 4y^2 - 1$$

We solve the system

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 0 \end{cases} \Longrightarrow \begin{cases} -y e^{-xy} = \lambda(2x) & \text{—— (1)} \\ -x e^{-xy} = \lambda(8y) & \text{—— (2)} \\ x^2 + 4y^2 - 1 = 0 & \text{—— (3)} \end{cases}$$

1

$(1) \Longrightarrow \underbrace{e^{-xy} = -\frac{2\lambda x}{y}}_{(4)}$ ($y \neq 0$: For if $y = 0$, Then (1) $\Rightarrow x = 0$. This gives $(x,y) = (0,0)$ which is not on the boundary.)

Sub. (4) in (2) : $-x \cdot \frac{-2\lambda x}{y} = 8\lambda y \Rightarrow 2\lambda x^2 = 8\lambda y^2$

$\Rightarrow 2\lambda(x^2 - 4y^2) = 0$

$\Rightarrow \lambda = 0$ or $x^2 = 4y^2$

- $\lambda = 0 \xRightarrow[(2)]{(1)} \cancel{(x,y)}\ x = 0, y = 0 \Rightarrow (x,y) = (0,0)$, rejected, as it does not satisfy (3)

1

- $x^2 = 4y^2 \xRightarrow{(3)} x^2 + x^2 - 1 = 0 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm \frac{1}{\sqrt{2}}$
  - $x = \frac{1}{\sqrt{2}} \Rightarrow \frac{1}{2} = 4y^2 \Rightarrow y^2 = \frac{1}{8} \Rightarrow y = \pm \frac{1}{2\sqrt{2}} \Rightarrow \boxed{\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right)}$
  - $x = \frac{-1}{\sqrt{2}} \Rightarrow \frac{1}{2} = 4y^2 \Rightarrow y^2 = \frac{1}{8} \Rightarrow y = \pm \frac{1}{2\sqrt{2}} \Rightarrow \boxed{\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right)}$

1

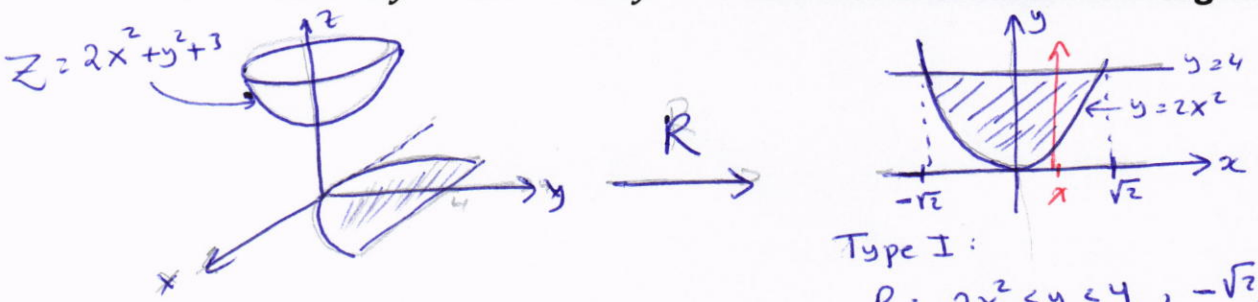
- $f(0,0) = 1$, $f\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = e^{-1/4}$, $f\left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) = e^{1/4}$, $f\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = e^{1/4}$, $f\left(-\frac{1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right) = e^{-1/4}$

min value of f is $e^{-1/4}$ & it occurs at $\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$ & $\left(-\frac{1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right)$

max —— $e^{1/4}$ —— $\left(\frac{1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right)$ & $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$

1

3. [4 points] Set up an integral for the volume of the solid that lies below the graph of $z = 2x^2 + y^2 + 3$ and above the region R bounded by the curves $y = 2x^2$ *and* $y = 4$. **Do not evaluate the integral.**



$2x^2=4$
$x^2=2$
$x=\pm\sqrt{2}$

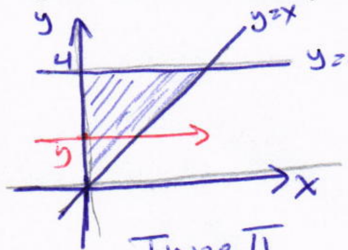
Type I:
$R: 2x^2 \le y \le 4, \ -\sqrt{2} \le x \le \sqrt{2}$

$$V = \iint_R f(x,y)\, dA \quad (1)$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \int_{2x^2}^{4} (2x^2+y^2+3)\, dy\, dx$$

(1) (1) (1)

4. [4 points] Evaluate $\iint_R y^2 e^{xy}\, dA$, where R is the region bounded by the curves $y = x, y = 4, x = 0$.



Type II

- It is easier to integrate $y^2 e^{xy}$ with respect to x first. So we choose $dA = dx\,dy$ (Type II)

$R: 0 \le x \le y, \ 0 \le y \le 4$

- $\iint_R y^2 e^{xy}\, dA$

$$= \int_0^4 \int_0^y y^2 e^{xy}\, dx\, dy \quad (2)$$

$$= \int_0^4 \int_0^y y \cdot y e^{xy}\, dx\, dy$$

$$= \int_0^4 y \cdot e^{xy}\Big]_{x=0}^{x=y}\, dy \quad (0.5)$$

$$= \int_0^4 y\,(e^{y^2}-1)\, dy$$

$$= \int_0^4 ye^{y^2} - y\, dy \quad (0.5)$$

$$= \frac{1}{2}e^{y^2} - \frac{1}{2}y^2\Big]_0^4 \quad (0.5)$$

$$= \left(\frac{1}{2}e^{16} - \frac{16}{2}\right) - \left(\frac{1}{2} - 0\right)$$

$$= \frac{1}{2}\left(e^{16} - 17\right). \quad (0.5)$$