Math201.05, Quiz #1, Term 161

Name:

ID #: Solutions

Serial #:

1. **[3 points]** Describe and sketch, with directions, the parametric curve given by

$$x = -\sqrt{t-1}, \qquad y = t+1, \qquad 1 \le t \le 10.$$

2. **[3 points]** Find the area of the surface generated by revolving the following parametric curve about the $x-axix$:

$$x = cos^3 t, \qquad y = sin^3 t, \qquad 0 \le t \le \pi/2.$$

3. **[4 points]** Find the area of the polar region that lies **inside both curves** $r = 1 + cos\theta$ and $r = 1 - cos\theta$.

Good luck,

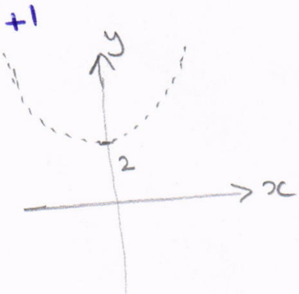
Ibrahim Al-Rasasi

1. Convert to a Cartesian equation:

$x = -\sqrt{t-1} \Rightarrow \sqrt{t-1} = -x \Rightarrow t-1 = x^2 \Rightarrow t = x^2+1$

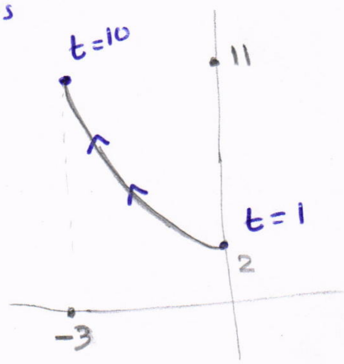
$\Rightarrow y = t+1 = x^2+1+1$

$\Rightarrow y = x^2+2$, a parabola 1.5 0.5



• For $1 \le t \le 10$, $x = -\sqrt{t-1} \le 0$ and $y = t+1 > 0$.

So we choose the part of the parabola that lies in the 2^{nd} quadrant.



• For the direction

$t=1 \Rightarrow (x,y) = (0,2)$, initial pt

$t=10 \Rightarrow (x,y) = (-3,11)$, terminal pt

2 $S = \int_0^{\pi/2} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

- $\frac{dx}{dt} = 3\cos^2 t\,(-\sin t)$; $\frac{dy}{dt} = 3\sin^2 t\,(\cos t)$
- $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t$

$= 9\cos^2 t \sin^2 t\, \underbrace{(\cos^2 t + \sin^2 t)}_{=1}$

$= 9\cos^2 t \sin^2 t$

- $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 3\,|\cos t \sin t|$

$= 3\cos t \sin t \quad , \quad 0 \le t \le \frac{\pi}{2}$

$$= 6\pi \int_0^{\pi/2} \sin^3 t \cdot \cos t \sin t \, dt$$

$$= 6\pi \int_0^{\pi/2} \sin^4 t \cdot \cos t \, dt \qquad : \quad u = \sin t \Rightarrow du = \cos t\, dt$$

- $t = 0 \Rightarrow u = 0$
- $t = \frac{\pi}{2} \Rightarrow u = 1$

$$= 6\pi \int_0^1 u^4\, du$$

$$= 6\pi \cdot \left.\frac{u^5}{5}\right]_0^1 = 6\pi\left(\frac{1}{5} - 0\right) = \frac{6\pi}{5} .$$

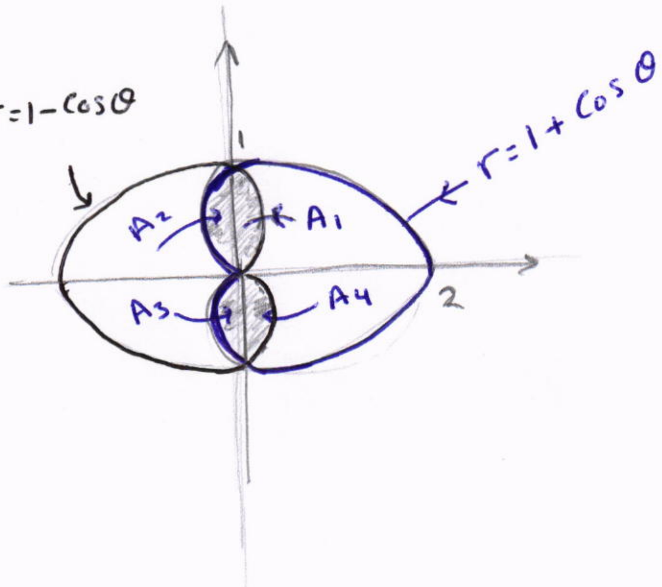
3
- points of intersection:

$1 + \cos\theta = 1 - \cos\theta$

$\Rightarrow 2\cos\theta = 0$

$\Rightarrow \cos\theta = 0$

$\Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$



- By Summetry, $A_1 = A_2 = A_3 = A_4$

- A_1 is the area of the region included by the curve $r = 1-\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$

$$A_1 = \int_0^{\pi/2} \frac{1}{2}(1-\cos\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 1 - 2\cos\theta + \cos^2\theta \; d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 1 - 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos(2\theta) \; d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos(2\theta) \; d\theta$$

$$= \frac{1}{2}\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin(2\theta)\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{3\pi}{4} - 2 + 0\right) - 0\right] = \frac{3\pi}{8} - 1$$

- The Total area $= 4A_1 = 4\left(\frac{3\pi}{8} - 1\right) = \frac{3\pi}{2} - 4.$

Math201.10, Quiz #1, Term 161

Name: Solutions

ID #: **Serial #:**

1. **[3 points]** Describe and sketch, with directions, the parametric curve given by

$$x = tant, \qquad y = -sect, \qquad 0 \le t < \pi/2.$$

2. **[3 points]** Find the slope of the tangent line to the polar curve $r = \sin(2\theta)$ at the point corresponding to $\theta = \pi/6$.
3. **[4 points]** Find the area of the polar region that lies **inside both curves** $r = 1$ and $r = 1 - sin\theta$.

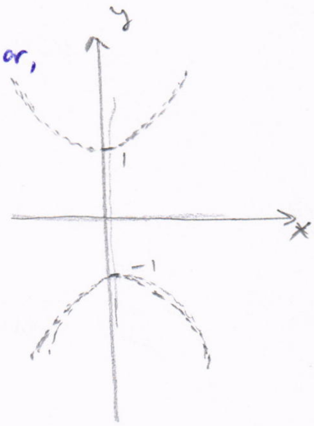
Good luck,

Ibrahim Al-Rasasi

$\boxed{1}$ • Convert to a Cartesian equation:

Since $1+\tan^2 t = \sec^2 t$, then we get $1+x^2 = y^2$, or,

$$y^2 - x^2 = 1, \text{ a hyperbola}$$



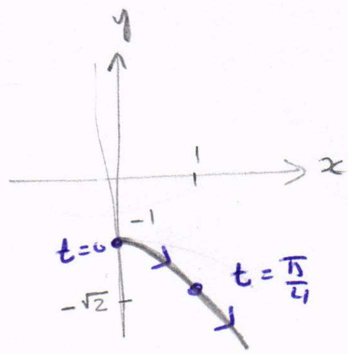
• For $0 \le t < \frac{\pi}{2}$, $x = \tan t \geq 0$ & $y = -\sec t < 0$.

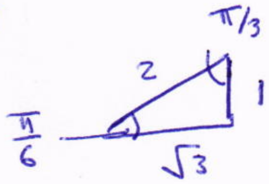
So we choose the part of the hyperbola that lies in the 4^{th} quadrant

• For the direction

$t = 0 \Rightarrow (x,y) = (0,-1)$, initial pt

$t = \frac{\pi}{4} \Rightarrow (x,y) = (1, -\sqrt{2})$





$\boxed{2}$ we parametrize the polar curve $r = \sin(2\theta)$:

- $x = r\cos\theta = \sin(2\theta)\cos\theta$
  $y = r\sin\theta = \sin(2\theta)\sin\theta$

- $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin(2\theta)\cdot\cos\theta + \sin\theta\cdot 2\cos(2\theta)}{\sin(2\theta)\cdot(-\sin\theta) + \cos\theta\cdot 2\cos(2\theta)}$$

- $$\text{slope} = \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{\sin(\frac{\pi}{3})\cdot\cos(\frac{\pi}{6}) + 2\cdot\sin(\frac{\pi}{6})\cos(\frac{\pi}{3})}{\sin(\frac{\pi}{3})\cdot(-\sin\frac{\pi}{6}) + 2\cos(\frac{\pi}{6})\cdot\cos(\frac{\pi}{3})}$$

$$= \frac{\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} + 2\cdot\frac{1}{2}\cdot\frac{1}{2}}{-\frac{\sqrt{3}}{2}\cdot\frac{1}{2} + 2\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2}}$$

$$= \frac{\frac{3}{4} + \frac{2}{4}}{\frac{\sqrt{3}}{4}} = \frac{\frac{5}{4}}{\frac{\sqrt{3}}{4}} = \frac{5}{\sqrt{3}}$$

<u>OR</u>, by the formula,

- $$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}, \quad f(\theta) = \sin(2\theta)$$

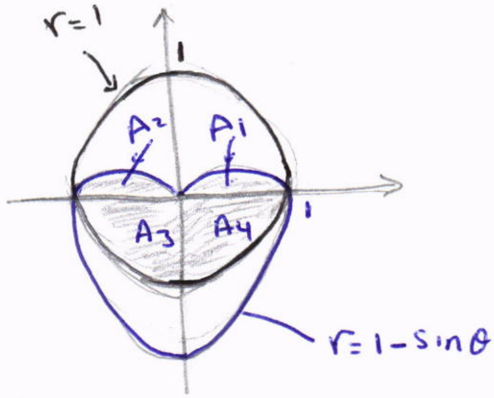
$$= \frac{2\cos(2\theta)\cdot\sin\theta + \sin(2\theta)\cdot\cos\theta}{2\cos(2\theta)\cos\theta - \sin(2\theta)\cdot\sin\theta}$$ (exactly as above)

- $$\text{slope} = \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \text{as above} = \frac{5}{\sqrt{3}}$$

[3] • points of intersection:

$1 = 1 - \sin\theta \implies \sin\theta = 0$

$\implies \theta = 0, \pi$



• By Symmetry,

$A_1 = A_2$

$A_3 = A_4$ = a quarter of the unit circle $(r=1)$

$= \frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$

• A_1 is the ~~area~~ of the region included by $r = 1 - \sin\theta$, for $0 \le \theta \le \frac{\pi}{2}$

$$A_1 = \int_0^{\pi/2} \frac{1}{2}(1-\sin\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 1 - 2\sin\theta + \sin^2\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 1 - 2\sin\theta + \frac{1}{2} - \frac{1}{2}\cos(2\theta) \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{3}{2} - 2\sin\theta - \frac{1}{2}\cos(2\theta) \, d\theta$$

$$= \frac{1}{2}\left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin(2\theta)\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{3\pi}{4} + 0 - 0\right) - (0 + 2 - 0)\right]$$

$$= \frac{3\pi}{8} - 1$$

• Total Area $= 2A_1 + 2A_3$

$$= 2\left(\frac{3\pi}{8} - 1\right) + 2\left(\frac{\pi}{4}\right)$$

$$= \frac{3\pi}{4} - 2 + \frac{2\pi}{4}$$

$$= \frac{5\pi}{4} - 2$$

Anoth method

OR, Let B be the area of the region lying inside the circle and outside the Cardioid. Then, by Symmetry,

$$B = 2 \cdot \int_0^{\pi/2} \frac{1}{2}\left[(1)^2 - (1-\sin\theta)^2\right] d\theta$$

$$= \int_0^{\pi/2} 1 - (1 - 2\sin\theta + \sin^2\theta) \quad d\theta$$

$$= \int_0^{\pi/2} 2\sin\theta - \sin^2\theta \quad d\theta$$

$$= \int_0^{\pi/2} 2\sin\theta - \frac{1}{2} + \frac{1}{2}\cos(2\theta) \quad d\theta$$

$$= \left. -2\cos\theta - \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) \right]_0^{\pi/2}$$

$$= \left(0 - \frac{\pi}{4} + 0\right) - (-2 - 0 + 0)$$

$$= -\frac{\pi}{4} + 2$$

Now,

The required area $=$ area of the unit circle $- B$

$$= \pi(1)^2 - \left(-\frac{\pi}{4} + 2\right)$$

$$= \pi + \frac{\pi}{4} - 2$$

$$= \frac{5\pi}{4} - 2.$$