King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 201 - Final Exam - Term 161

Time Allowed: 180 minutes

MASTER

Key

Name: ______ ID Number: ______

Section Number: ______ Serial Number: ______

Class Time: ______ Instructor's Name: ______

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification
4. Make sure that you have 12 pages of problems (Total of 21 Problems)

Question	Points	Out of
1		10
2		10
3		10
4		10
5		10
6		10
7		10
Written		70
MCQ		70
Total		140

1. Find the local maximum and minimum value(s) and saddle point(s) of the function

$$f(x,y) = x^3y + 24x^2 + 8y^2.$$

(2)

$f_x = 3x^2y + 48x$, $\quad f_y = x^3 + 16y$

$f_{xx} = 6xy + 48 \qquad f_{yy} = 16 \qquad f_{xy} = 3x^2$

Critical points

$3x^2y + 48x = 0$ —— (1)

$x^3 + 16y = 0$ —— (2)

(4) From (1)

$x = 0$

Sub. in (2)

$y = 0$

or

$y = -\frac{16}{x}$

sub in (2)

$x^3 - \frac{256}{x} = 0$

$x = \pm 4$

$y = \mp 4$

(4)

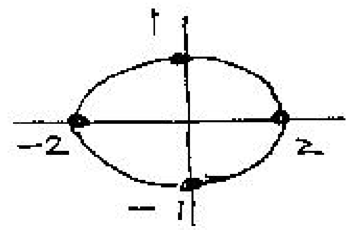
Point	f_{xx}	f_{yy}	f_{xy}	D		value
(0,0)	48	16	0	>0	local min	0
(4,-4)	-48	16	48	<0	saddle point	
(-4,4)	-48	16	48	<0	saddle point	

2. Use Lagrange multipliers to find the maximum and minimum values of the function $f(x,y) = y^2 - x^2$ on the ellipse $\frac{x^2}{4} + y^2 = 1$.

Let $g(x,y) = \frac{x^2}{4} + y^2 - 1$

1 $\nabla f = \langle -2x, 2y \rangle$, $\nabla g = \langle \frac{x}{2}, 2y \rangle$

Critical points on the ellipse



$\nabla f = \lambda \nabla g$, $g = 0$

$-2x = \lambda \frac{x}{2}$, ——— (1)

3 $y = \lambda y$, ——— (2)

$\frac{x^2}{4} + y^2 = 1$ ——— (3)

From (1)

$x = 0$ or $\lambda = -4$

From (3)

3 $y = \pm 1$

From (2) $\lambda = 1$

$(0,1)$, $(0,-1)$

From (2)

$y = 0$

From (3)

$x = \pm 2$

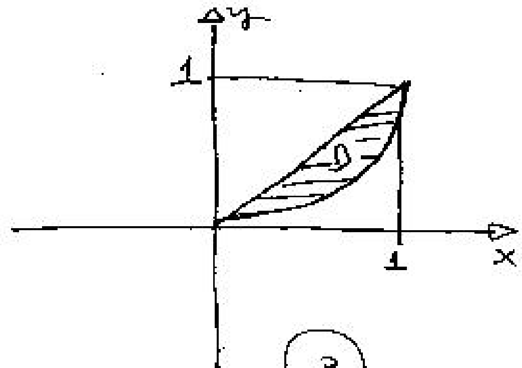
$(-2,0)$, $(2,0)$

2

(x,y)	$f(x,y)$
$(2,0)$	-4
$(0,1)$	1
$(-2,0)$	-4
$(0,-1)$	1

1 $\therefore f_{max} = 1$, $f_{min} = -4$

3. Sketch the region D bounded by $y = x$, $y = x^3$, $x \geq 0$ and then use it to evaluate the double integral $\iint_D (x^2 + 2y)\, dA$.



3

$$\iint_D (x^2 + 2y)\, dA$$

3 $$= \int_0^1 \int_{x^3}^{x} (x^2 + 2y)\, dy\, dx$$

1 $$= \int_0^1 x^2 y + y^2 \Big]_{x^3}^{x} dx$$

1 $$= \int_0^1 (x^3 + x^2 - x^5 - x^6)\, dx$$

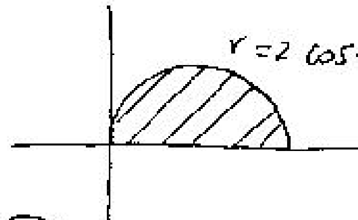
1 $$= \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^6}{6} - \frac{x^7}{7} \Big]_0^1$$

1 $$= \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{7} = \frac{23}{84}$$

4. Evaluate the iterated integral $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

Change to polar coordinates:

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$



(3)

$y^2 = 2x - x^2,\ y \geq 0$

$\Rightarrow r^2 = 2r\cos\theta$

$\Rightarrow r = 2\cos\theta,$

$0 \leq \theta \leq \frac{\pi}{2}$

(4) $$= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{8}{3}\cos^3\theta\, d\theta = \frac{8}{3}\int_0^{\frac{\pi}{2}} (1-\sin^2\theta)\cos\theta\, d\theta$$

(3) $$= \frac{8}{3}\left[\sin\theta - \frac{\sin^3\theta}{3}\right]_0^{\frac{\pi}{2}} = \frac{8}{3}\left(1-\frac{1}{3}\right)$$

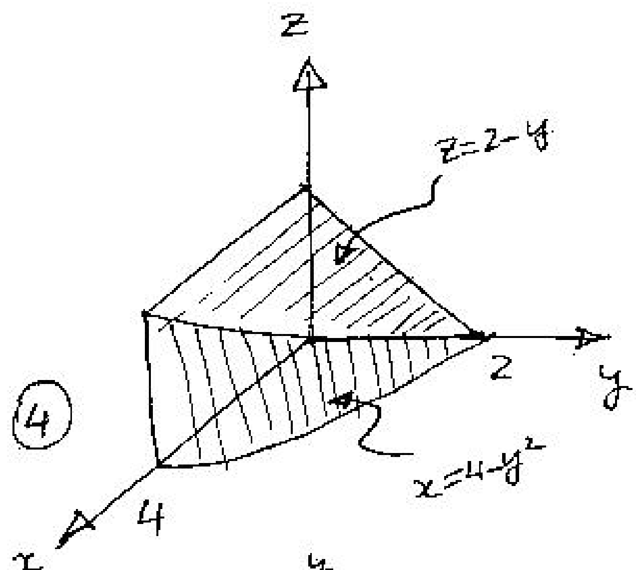
$$= \frac{16}{3}$$

5. Sketch the solid indicated by the iterated integral $\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\, dz\, dy$ and then write it down in the order $dz\, dy\, dx$.

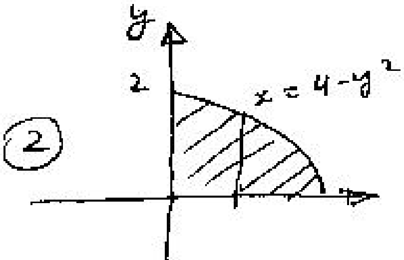
$$\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\, dz\, dy$$

(4) $$= \int_0^4 \int_0^{\sqrt{4-x}} \int_0^{2-y} dz\, dy\, dx$$

(4)



(2)



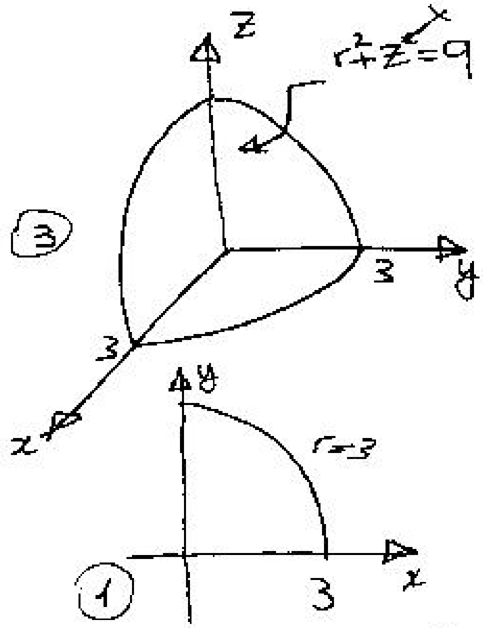
Projection in the xy-plane

6. Change the integral $\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$ to one in cylindrical coordinates. Do not evaluate the integral.

$$\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$$

$$=\int_0^{\frac{\pi}{2}}\int_0^3\int_0^{\sqrt{9-r^2}} r^2\,dz\,dr\,d\theta$$

1 1 1 1 2



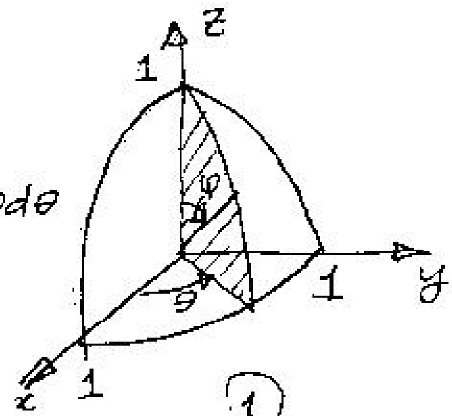
projection in the $r\theta$-plane

7. Evaluate $\iiint_E x e^{\sqrt{x^2+y^2+z^2}}\,dV$, where E is the portion of the unit ball $x^2+y^2+z^2\le 1$ that lies in the first octant.

$$\iiint_E x e^{\sqrt{x^2+y^2+z^2}}\,dV$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \rho\sin\varphi\cos\theta\; e^{\rho}\;\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

1 1 1 1 1 2



$$=\int_0^1\rho^3 e^{\rho}\,d\rho\int_0^{\frac{\pi}{2}}\sin^2\varphi\,d\varphi\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta$$

2

$$=(\rho^3-3\rho^2+6\rho-6)e^{\rho}\Big]_0^1\int_0^{\frac{\pi}{2}}\frac{1}{2}(1-\cos\varphi)\,d\varphi$$

$$=(-2e+6)\frac{\pi}{4}=\frac{\pi}{2}(3-e).$$

8. The graph of the curve $r = 1 - 2\cos\theta$ has a vertical tangent at

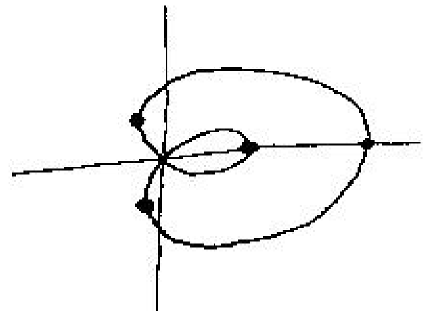
a) 4 points

b) 2 points

c) 3 points

d) one point

e) 5 points



4 points

9. The area inside the outer loop of the curve $r = 1 + 2\cos\theta$ is

a) $2\pi + \dfrac{3\sqrt{3}}{2}$

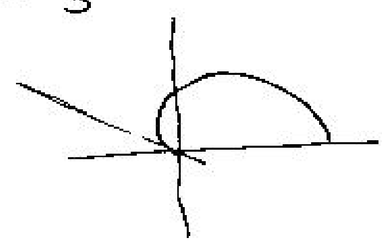
b) $\pi + 3\sqrt{3}$

--> c) $2\pi -$ [illegible]

d) $\pi + 3$

e) $\pi - \dfrac{3\sqrt{3}}{2}$

$1 + 2\cos\theta = 0 \qquad \theta = \dfrac{2\pi}{3}$



$$\int_0^{\frac{2\pi}{3}} (1 + 2\cos\theta)^2 \, d\theta$$

$$= 2\pi + \frac{3\sqrt{3}}{2}$$

10. The length of the parametric curve $x = a\cos^3 t$, $y = a\sin^3 t$, $0 \le t \le \frac{\pi}{4}$ is

a) $\frac{3a}{4}$

b) $3a$

c) $\frac{3a^2}{4}$

d) $3a^2$

e) $\frac{3a}{5}$

$\frac{dx}{dt} = -3a\cos^2 t \sin t \qquad \frac{dy}{dt} = 3a\sin^2 t \cos t$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9a^2\cos^2 t \sin^2 t$

$L = \int_0^{\frac{\pi}{4}} 3a\cos t \sin t = \frac{3a}{2}\sin^2 t \Big|_0^{\frac{\pi}{4}}$

$= \frac{3a}{2} * \frac{1}{2} = \frac{3a}{4}$

11. The distance from the point $P(1, 2, -1)$ to the plane $3x - 2y + 6z = 5$ is

a) $\frac{12}{7}$

b) $\frac{7}{12}$

c) 12

d) 7

e) 5

$d = \frac{|3 - 4 - 6 - 5|}{\sqrt{9 + 4 + 36}}$

$= \frac{12}{7}$

12. Let $\mathbf{a} = \langle 1, 2, -4\rangle$, $\mathbf{b} = \langle 3, 1, 1\rangle$. A vector parallel to $\mathbf{a} \times \mathbf{b}$ is

a) $\langle -30, 65, 25\rangle$

b) $\langle 12, -13, 10\rangle$

c) $\langle 1, -10, 10\rangle$

d) $\langle 14, 65, 20\rangle$

e) $\langle 12, 13, 25\rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -4 \\ 3 & 1 & 1 \end{vmatrix}$$

$$= \langle 6, -13, -5\rangle$$

$$= -\frac{1}{5}\langle -30, 65, 25\rangle$$

13. The surface $y^2 = x^2 + \frac{1}{9}z^2$ is

a) an elliptic cone with axis along the y-axis

b) an elliptic cone with axis along the z-axis

c) an elliptic paraboloid with axis along the y-axis

d) an elliptic paraboloid with axis along the z-axis

e) an elliptic paraboloid with axis along the x-axis

14. The domain of the function $f(x,y) = \sqrt{y} + \sqrt{25 - x^2 - y^2}$ is

a) the closed upper half of the disc $x^2 + y^2 \leq 25$

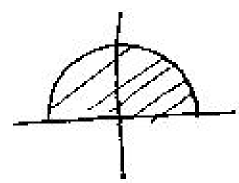
b) the closed upper half of the disc $x^2 + y^2 \leq 25$ except the point $(0,0)$

c) the closed disc $x^2 + y^2 \leq 25$ except the y-axis

d) the closed upper half of the disc $x^2 + y^2 \leq 25$ except the x-axis

e) the closed right half of the disc $x^2 + y^2 \leq 25$

$y \geq 0$, $x^2 + y^2 \leq 25$



15. Let $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. Then f is not continuous at $(0,0)$ because

a) $\lim_{(x,y) \to (0,0)} f(x,y)$ does not exist

b) $f(0,0) \neq 1$

c) f is not defined at $(0,0)$

d) $\lim_{(x,y) \to (0,0)} f(x,y)$ exists but not equal $f(0,0)$

e) $f(x,y)$ is not defined around $(0,0)$

18. The directional derivative of the function $g(p,q) = p^4 - p^2q^3$ at the point $(2,1)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$ is

a) $-\frac{8}{\sqrt{10}}$

b) $\frac{23}{\sqrt{10}}$

c) $-\frac{22}{\sqrt{10}}$

d) $\frac{24}{\sqrt{10}}$

e) $\frac{4}{\sqrt{10}}$

$\nabla g = \langle 4p^3 - 2pq^3, -3p^2q^2 \rangle$

$\nabla g(2,1) = \langle 28, -12 \rangle$

$\vec{u}_{\vec{v}} = \langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \rangle$

$\nabla g \cdot \vec{u}_{\vec{v}} = \frac{28 - 36}{\sqrt{10}} = -\frac{8}{\sqrt{10}}$

19. The equation of the tangent plane to the surface $xy + yz + zx = 5$ at the point $(1,2,1)$ is

a) $3x + 2y + 3z = 10$

b) $x + 2y + 3z = 8$

c) $3x + y + 3z = 8$

d) $3x - 2y + 3z = 2$

e) $x - y - z = 0$

$\nabla F = \langle y+z, z+x, x+y \rangle$

$\nabla F(1,2,1) = \langle 3,2,3 \rangle$

$3(x-1) + 2(y-2) + 3(z-1) = 0$

$3x + 2y + 3z = 10$

16. If $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$, then $\frac{\partial R}{\partial R_1}$ equals

a) $\frac{R^2}{R_1^2}$

b) $-\frac{R^2}{R_2^2 + R_3^2}$

c) $\frac{1}{R^2} + \frac{1}{R_1^2}$

d) $R^2 + \frac{1}{R_1^2}$

e) $\frac{R_2^2 + R_3^2}{R_1^2}$

Taking Partial derivatives on both sides wrt R_1:

$$-\frac{1}{R^2}\frac{\partial R}{\partial R_1} = -\frac{1}{R_1^2}$$

$$\frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2}$$

17. Using linear approximation to estimate $\sqrt{(3.02)^2 + (3.98)^2}$ gives

a) 4.996

b) 4.96

c) 5.002

d) 5.02

e) 5.004

Let $f(x,y) = \sqrt{x^2+y^2}$, $(x_0,y_0) = (3,4)$

$$f_x = \frac{x}{\sqrt{x^2+y^2}} \qquad f_y = \frac{y}{\sqrt{x^2+y^2}}$$

$f(3,4) = 5$, $f_x(3,4) = \frac{3}{5}$,

$f_y(3,4) = \frac{3}{5}$

$$L(x,y) = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4)$$

$$L(3.02, 3.98) = 5 + \frac{.06}{5} - \frac{.08}{5}$$

$$= 4.996$$

20. If the temperature at any point (x, y) is given by $T(x, y) = 200\, e^{-x^2-3y^2}$, then the temperature increases most rapidly at the point $(2, -1)$ in the direction

a) $\langle -1, 6\rangle$

b) $\langle 200\, e^{-1}, 200\, e^{-2}\rangle$

c) $\langle e^{-3}, 62\, e^{-1}\rangle$

d) $\langle 8, -12\rangle$

e) $\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle$

$\nabla T \sim \langle -2x, -6y\rangle$

at $(2, -1)$

$\langle -4, 6\rangle$

21. If $u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, then $u_{x_1}^2 + u_{x_2}^2 + \cdots + u_{x_n}^2$ simplifies to

a) 1

b) -1

c) u

d) $-u$

e) u^2

$u_{x_i} = \frac{u_i}{u}$

$\sum u_{x_i}^2 = \frac{\sum u_i^2}{u^2}$

$= \frac{u^2}{u^2}$

$= 1$