King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 102 - Exam II - Term 161

Duration: 100 minutes

Name: Key

ID Number: ☺

Section Number: __________

Serial Number: __________

Class Time: __________

Instructor's Name: __________

Instructions:

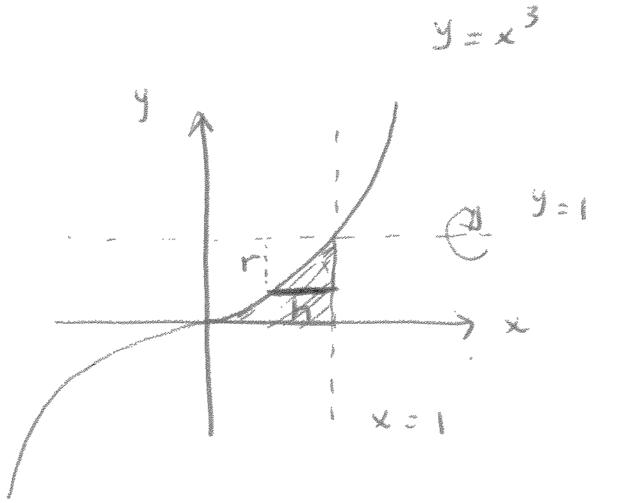
1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 10 pages of problems (Total of 10 Problems)

Question Number	Points	Maximum Points
1		10
2		10
3		10
4		10
5		12
6		10
7		10
8		10
9		10
10		8
Total		100

1. **[10 points]** Use the method of Cylindrical Shells to find the volume of the solid generated by revolving the region bounded by the curves

$$y = x^3, \ y = 0, \ \text{and } x = 1,$$

about the line $y = 1$.



Ans.

$r = 1 - y$ 1pt

$h = 1 - x = 1 - \sqrt[3]{y}$ 2pts

$$V = \int_0^1 2\pi(1-y)(1-\sqrt[3]{y})\,dy \quad \text{3pts}$$

$$= 2\pi \int_0^1 (1 - y^{1/3} - y + y^{4/3})\,dy \quad \text{1pt}$$

$$= 2\pi \left(y - \frac{3}{4} y^{4/3} - \frac{1}{2} y^2 + \frac{3}{7} y^{7/3} \right) \Big|_0^1 \quad \text{2pts}$$

$$= 2\pi \left(\frac{5}{28} \right) = \frac{5\pi}{14} \quad \text{1pt}$$

2. **[10 points]** Use a suitable substitution to calculate the integral

$$\int_0^{\pi/3} \frac{1-\cos x}{\cos^2 x} \sin x \, dx.$$

Ans Let $u = \cos x$. Then, $du = -\sin x \, dx$ 2pts

$x = 0 \Rightarrow u = 1$
$x = \frac{\pi}{3} \Rightarrow u = \frac{1}{2}$ 1pt

$$\int_0^{\pi/3} \frac{1-\cos x}{\cos^2 x} \sin x \, dx = -\int_1^{1/2} \frac{1-u}{u^2} \, du$$ 2pts

$$= \int_{1/2}^{1} \left(u^{-2} - \frac{1}{u}\right) du$$ 1pt

$$= -u^{-1} - \ln|u| \Big|_{1/2}^{1}$$ 2pts

$$= -1 - \ln 1 - (-2 - \ln 1/2)$$

$$= 1 - \ln 2$$ 2pts

3. **[10 points]** Evaluate the integral

$$\int \sec^7 x \tan^3 x \, dx.$$

Ans:

$I = \int \sec^7 x \tan^3 x \, dx = \int \sec^6 x \tan^2 x \cdot \sec x \tan x \, dx$. 2pts

$= \int \sec^6 x (\sec^2 x - 1) \sec x \tan x \, dx$ 1pt

let $u = \sec x \Rightarrow du = \sec x \tan x \, dx$. 2pts

Then, $I = \int u^6 (u^2 - 1) \, du$

$= \int (u^8 - u^6) \, du$ 2pts

$= \frac{1}{9} u^9 - \frac{1}{7} u^7 + C$ 2pts

$= \frac{1}{9} \sec^9 - \frac{1}{7} \sec^7 x + C$. 1pt

4. **[10 points]** Use Integration by Parts to evaluate the integral

$$\int_{-1}^{1} \cot^{-1} x \, dx.$$

Ans. let $u = \cot^{-1} x$, $dv = dx$

$du = -\frac{1}{1+x^2} dx$, $V = x$ 4pts

Then, $\int_{-1}^{1} \cot^{-1} x \, dx = x \cot^{-1} x \Big|_{-1}^{1} + \int_{-1}^{1} \frac{x}{x^2+1} dx$ 2pts

$= x \cot^{-1} x \Big|_{-1}^{1}$ 2pts

$= 1 \cdot \frac{\pi}{4} + 1 \cdot \frac{3\pi}{4} = \pi$ 2pts

5. **[12 points]** Use the substitution $t = \ln x$ to calculate the integral

$$\int \sin(\ln x)\, dx.$$

Ans. $t = \ln x \Rightarrow dt = \frac{1}{x} dx$. Note, $x = e^t$. 2pts

$I = \int \sin(\ln x)\, dx = \int \sin t \cdot e^t\, dt$ 2pts

$\begin{cases} u = e^t, & dv = \sin t\, dt \\ du = e^t\, dt, & v = -\cos t \end{cases}$

$= -e^t \cos t + \int e^t \cos t\, dt$ 2pts

$\begin{cases} u = e^t, & dv = \cos t\, dt \\ du = e^t\, dt, & v = \sin t \end{cases}$

$= -e^t \cos t + e^t \sin t - \underbrace{\int e^t \sin t\, dt}_{I}$ 2pts

$\Rightarrow 2I = -e^t \cos t + e^t \sin t + 2C$

$\Rightarrow I = \frac{1}{2} e^t (\sin t - \cos t) + C$ 2pts

$= \frac{1}{2} x \left[\sin(\ln x) - \cos(\ln x)\right] + C.$ 2pts

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6. **[10 points]** Evaluate

$$\int \frac{x^2}{\sqrt{8-2x-x^2}}\,dx.$$

Ans: Note, $8-2x-x^2 = 9-(x+1)^2$.

Then, $I = \int \frac{x^2}{\sqrt{8-2x-x^2}}\,dx = \int \frac{x^2}{\sqrt{9-(x+1)^2}}\,dx$. 1 pt

Let $\boxed{x+1 = 3\sin\theta}$. Then, $dx = 3\cos\theta\, d\theta$ 2pts.

$I = \int \frac{(3\sin\theta-1)^2}{3\cos\theta}\, 3\cos\theta\, d\theta$ 2pts

$\sqrt{9-(x+1)^2} = \sqrt{9(1-\sin^2\theta)} = 3\cos\theta$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$= \int (9\sin^2\theta - 6\sin\theta + 1)\, d\theta$ 1pt

$= \int \left[\frac{9}{2}(1-\cos 2\theta) - 6\sin\theta + 1\right] d\theta$ 2pts

$= \frac{11}{2}\theta - \frac{9}{4}\sin 2\theta + 6\cos\theta + C$

$= \frac{11}{2}\theta - \frac{9}{2}\sin\theta\cos\theta + 6\cos\theta + C$

$= \frac{11}{2}\sin^{-1}\left(\frac{x+1}{3}\right) - \frac{9}{2}\cdot\frac{x+1}{3}\cdot\frac{\sqrt{9-(x+1)^2}}{3} + 6\,\frac{\sqrt{9-(x+1)^2}}{3} + C$

$= \frac{11}{2}\sin^{-1}\left(\frac{x+1}{3}\right) - \frac{1}{2}(x+1)\sqrt{9-(x+1)^2} + 2\sqrt{9-(x+1)^2} + C$ 2pts

or $= \frac{11}{2}\sin^{-1}\left(\frac{x+1}{3}\right) + \frac{1}{2}(3-x)\sqrt{9-(x+1)^2} + C$.

7. **[10 points]** Use Partial Fractions to evaluate the integral

$$\int \frac{x^3 + 2x^2 + 5x + 4}{x^3 + 4x}\, dx.$$

Ans: Write $\frac{x^3+2x^2+5x+4}{x^3+4x} = 1 + \frac{2x^2+x+4}{x(x^2+4)}$, 2pts

and $\frac{2x^2+x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$ 2pts

$\Rightarrow 2x^2+x+4 = A(x^2+4) + x(Bx+C)$

$\Rightarrow 2x^2+x+4 = (A+B)x^2 + Cx + 4A$

Then, $A=1$, $C=1$, and $A+B=2 \Rightarrow B=1$. 2pts

$$\int \frac{x^3+2x^2+5x+4}{x^3+4x}\,dx = \int \left(1+\frac{1}{x}+\frac{x+1}{x^2+4}\right)dx$$

$$= \int \left(1+\frac{1}{x}\right)dx + \frac{1}{2}\int \frac{2x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx$$

$$= x + \ln|x| + \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C.$$

4pts

8. **[10 points]** Determine the value of the improper integral

$$\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx.$$

Ans. $\underline{I} = \int_0^\infty \frac{e^x}{e^{2x}+1}\,dx = \lim_{t\to\infty} \int_0^t \frac{e^x}{e^{2x}+1}\,dx.$ 2pts

$$\int_0^t \frac{e^x}{e^{2x}+1}\,dx = \int_1^{e^t} \frac{du}{u^2+1},$$ 2pts

$u = e^x \Rightarrow du = e^x dx$
$x = 0,\ u = 1$
$x = t,\ u = e^t$
2pts

$$= \tan^{-1} u \Big|_1^{e^t}$$

$$= \tan^{-1} e^t - \frac{\pi}{4}$$ 2pts

Then, $\underline{I} = \lim_{t\to\infty} \left(\tan^{-1} e^t - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ 2pts

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9. **[10 points]** Find the arc length function of the curve $y = x^2 - \frac{1}{8}\ln x$ taking $P(1,1)$ as a starting point.

Ans: Let $f(t) = t^2 - \frac{1}{8}\ln t$. Then,

$$L(x) = \int_1^x \sqrt{1+(f'(t))^2}\,dt \qquad \underline{2pts} \qquad \boxed{f'(t) = 2t - \frac{1}{8t}}$$

$$= \int_1^x \sqrt{1+\left(2t-\frac{1}{8t}\right)^2}\,dt \qquad \underline{2pts}$$

$$= \int_1^x \sqrt{\left(2t+\frac{1}{8t}\right)^2}\,dt$$

$$= \int_1^x \left(2t+\frac{1}{8t}\right)dt \qquad \underline{3pts}$$

$$= t^2 + \frac{1}{8}\ln t \Big|_1^x \qquad \underline{2pts}$$

$$= x^2 + \frac{1}{8}\ln x - 1 \qquad \underline{1pt}$$

10. **[8 points]** Find the average value of $f(x) = \cos^2(\pi x)$ on the interval $[-1, 1]$.

Ans. $\text{Average} = \frac{1}{2}\int_{-1}^{1} \cos^2(\pi x)\,dx$ 3pts

$= \frac{1}{4}\int_{-1}^{1} (1 + \cos 2\pi x)\,dx$ 2pts

$= \frac{1}{4}\left(x + \frac{1}{2\pi}\sin 2\pi x\right)\Big|_{-1}^{1}$ 2pts

$= \frac{1}{4}\cdot 2 = \frac{1}{2}$ 1pt