King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 101 - Exam I - Term 161

Duration: 90 minutes

Name: Key (revised) ID Number:

Section Number: Serial Number:

Class Time: Instructor's Name:

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 7 pages of problems (Total of 7 Problems)

Question Number	Points	Maximum Points
1		18
2		10
3		14
4		16
5		10
6		16
7		16
Total		100

1. Evaluate the following limits, if exist.

(a)[**6 points**] $\lim_{x\to 2} \frac{\sqrt{4x+1}-3}{x-2}$

$$\lim_{x\to 2} \frac{\sqrt{4x+1}-3}{x-2} = \lim_{x\to 2} \frac{\sqrt{4x+1}-3}{x-2} \cdot \frac{\sqrt{4x+1}+3}{\sqrt{4x+1}+3} \quad (2\text{ pts})$$

$$= \lim_{x\to 2} \frac{(4x+1)-9}{(x-2)(\sqrt{4x+1}+3)} \quad (2\text{ pts})$$

$$= \lim_{x\to 2} \frac{4\cancel{(x-2)}}{\cancel{(x-2)}(\sqrt{4x+1}+3)} \quad (1\text{ pt})$$

$$= \lim_{x\to 2} \frac{4}{\sqrt{4x+1}+3} = \frac{4}{6} = \frac{2}{3} \quad (1\text{ pt})$$

(b)[**6 points**] $\lim_{x\to -6} \frac{2x+12}{|x+6|}$

$$\lim_{x\to -6^-} \frac{2x+12}{|x+6|} = \lim_{x\to -6^-} \frac{2(x+6)}{-(x+6)} \quad (1\text{ pt})$$

$$= \lim_{x\to -6^-} -2 = -2, \quad (1\text{ pt})$$

$$\lim_{x\to -6^+} \frac{2x+12}{|x+6|} = \lim_{x\to -6^+} \frac{2\cancel{(x+6)}}{\cancel{(x+6)}} \quad (1\text{ pt})$$

$$= \lim_{x\to -6^+} 2$$

$$= 2 \quad (1\text{ pt})$$

Since $\lim_{x\to -6^-} \frac{2x+12}{|x+6|} \neq \lim_{x\to -6^+} \frac{2x+12}{|x+6|}$

$\therefore \lim_{x\to -6} \frac{2x+12}{|x+6|}$ DNE. (2pts)

(c) [**6 points**] $\lim_{x\to 0} x^6 \sin\left(\frac{\pi}{x}\right)$

$$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1 \quad , x \neq 0 \quad (1\text{ pt})$$

$$\Rightarrow -x^6 \leq x^6 \sin\left(\frac{\pi}{x}\right) \leq x^6 \quad (1\text{ pt})$$

Since $\lim_{x\to 0} x^6 = \lim_{x\to 0} -x^6 = 0$, (2 pts)

by sequeeze theorem $\lim_{x\to 0} x^6 \sin\left(\frac{\pi}{x}\right) = 0$ (1pt)

(1 pt)

2. **[10 points]** Find the values of a and b making the function

$$f(x) = \begin{cases} [|2x|] & \text{if } \quad x < -1 \\ ax+b & \text{if } \quad -1 \le x < 1 \\ 1-x^2 & \text{if } \quad x \ge 1 \end{cases}$$ continuous at $x=1$ and $x=-1$.

(where $[|2x|]$ is the greatest integer less than or equal to $2x$).

at $x=-1$: we need $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(-1)$ (1pt)

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} [\![2x]\!] = -3$ (1pt)

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (ax+b) = -a+b$ (1pt)

$f(-1) = -a+b$ (1pt)

$\therefore \quad -a+b = -3 \quad \ldots (1)$

at $x=1$: we need $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$ (1pt)

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (ax+b) = a+b$ (1pt)

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (1-x^2) = 0$ (1pt)

$f(1) = 0.$ (1pt)

$\therefore \quad a+b=0 \quad \ldots (2)$

from (1) and (2)

$2b = -3 \Rightarrow b = \frac{-3}{2}$, (1pt)

$\Rightarrow a = \frac{3}{2}$ (1pt)

3. (a)[**8 points**] Show that the equation $e^{-x+1} = \ln(x+e)$ has a solution in the interval $(0, 1)$.

let $f(x) = e^{-x+1} - \ln(x+e)$. (2 pts)

Since e^{-x+1} and $\ln(x+e)$ are both continuous on $[0,1]$

$\Rightarrow f(x)$ is continuous on $[0,1]$. (2 pts)

$f(0) = e - 1 > 0$, (1 pt)

$f(1) = 1 - \ln(1+e) < 0$, (1 pt)

So, by Intermediate Value Theorem there is

(2 pts) a number $c \in (0,1)$ such that $f(c) = 0$

$\Leftrightarrow e^{-c+1} - \ln(c+e) = 0 \Leftrightarrow e^{-c+1} = \ln(c+e)$

(b)[**6 points**] Evaluate the limit if it exists

$$\lim_{x\to 3} \cos^{-1}\left(\frac{2x-6}{x^2-2x-3}\right)$$

$$\lim_{x\to 3} \frac{2x-6}{x^2-2x-3} = \lim_{x\to 3} \frac{2(x-3)}{(x-3)(x+1)} \quad (2\text{ pts})$$

$$= \lim_{x\to 3} \frac{2}{x+1} = \frac{1}{2} \quad (1\text{ pt})$$

and since $\cos^{-1} x$ is continuous at $\frac{1}{2}$,

then

$$\lim_{x\to 3} \cos^{-1}\left(\frac{2x-6}{x^2-2x-3}\right) = \cos^{-1}\left(\lim_{x\to 3}\frac{2x-6}{x^2-2x-3}\right) \quad (1\text{ pt})$$

$$= \cos^{-1}\left(\frac{1}{2}\right) \quad (1\text{ pt})$$

$$= \frac{\pi}{3} \quad (1\text{ pt})$$

1. (a) **[8 points]** Find all vertical asymptote(s) of the function

$$f(x) = \frac{x^2 - 1}{2x^2 - 5x + 3}$$

(Justify your answer using limits)

(2pts)

$$f(x) = \frac{x^2-1}{2x^2-5x+3} = \frac{(x-1)(x+1)}{(2x-3)(x-1)} \quad , \quad x \neq \frac{3}{2}, x \neq 1$$

$$= \frac{x+1}{2x-3} \quad (1\text{ pt})$$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x+1}{2x-3} = -2$$

$\therefore$ $x=1$ is not a vertical asymptote

$$\lim_{x\to \frac{3}{2}} |f(x)| = \lim_{x\to\frac{3}{2}} \left|\frac{x+1}{2x-3}\right| = \infty \quad (4\text{pts})$$

$\therefore$ $x = \frac{3}{2}$ is a vertical asymptote. (2pts)

(b) **[8 points]** Find all horizontal asymptotes of $f(x) = \frac{1-e^x}{1+2e^x}$

(Justify your answer using limits)

$$(1\text{pt}) \quad \lim_{x\to -\infty} \frac{1-e^x}{1+2e^x} = \frac{1}{1} = 1 \quad (1\text{pt})$$

$\Rightarrow y = 1$ is H.A. (1pt)

$$(1\text{pt}) \quad \lim_{x\to\infty} \frac{1-e^x}{1+2e^x} = \lim_{x\to\infty} \frac{e^{-x}-1}{e^{-x}+2} \quad (2\text{pts})$$

$$= -\frac{1}{2} \quad (1\text{pt})$$

$\Rightarrow y = -\frac{1}{2}$ is another H.A. (1pt)

7. The displacement (in meters) of a particle moving in a straight line is given by $S = t^2 + 1$, where t is measured in seconds.

(a)**[6 points]** Find the average velocity over the time interval $[2, 3]$.

$$V_{ave} = \frac{S(3) - S(2)}{3-2} \quad (3pts)$$

$$= \frac{10-5}{1} = 5 \quad m/s \quad (3pts)$$

(b)**[10 points]** Sketch the graph of a function that satisfies all of the following:

(2pts) (i) $\lim_{x \to -\infty} f(x) = 0$,

(2pts) (ii) f has a jump discontinuity at $x = -2$

(1pt) (iii) $f(-2) = -1$

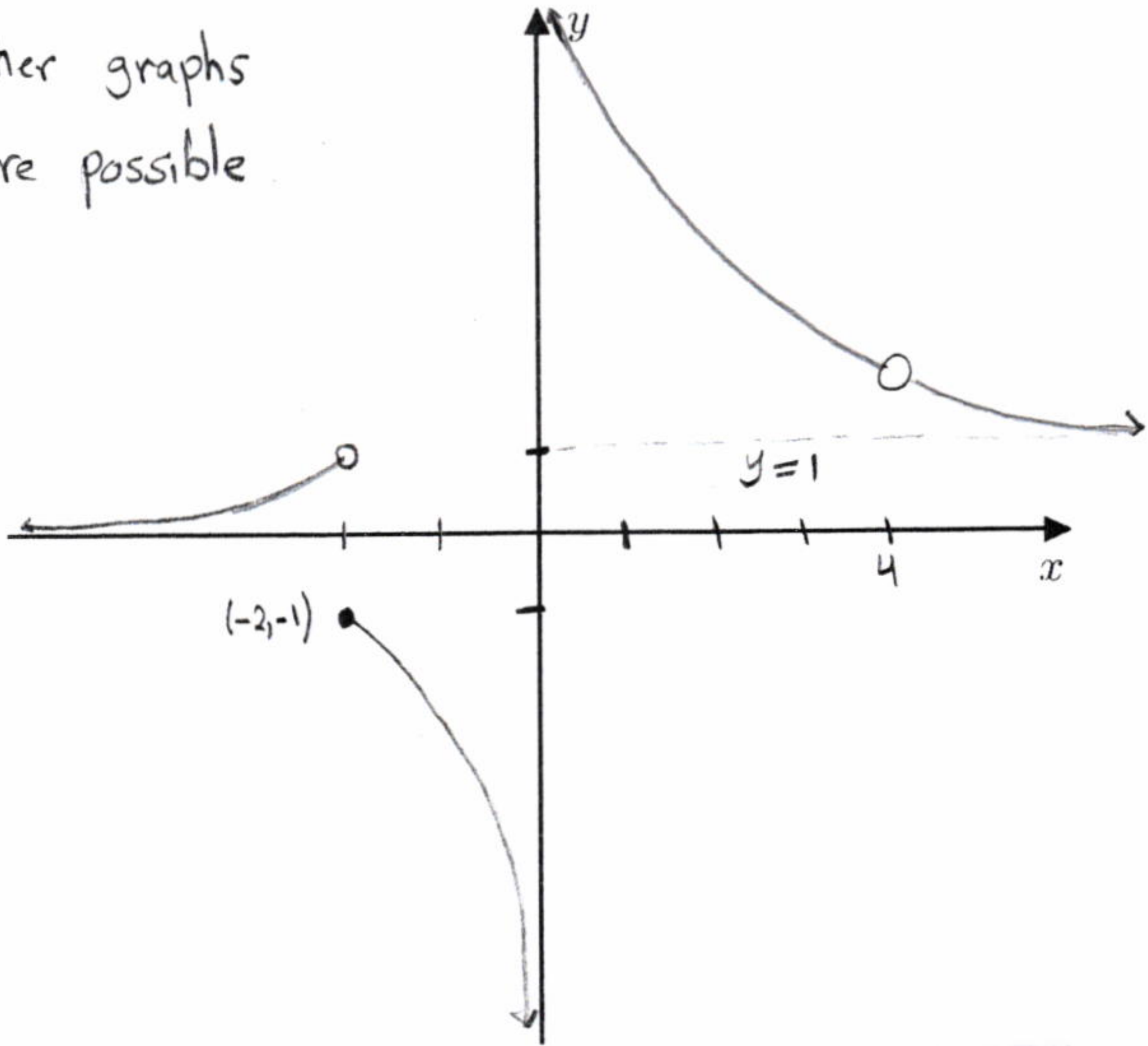
(1pt) (iv) $\lim_{x \to 0^-} f(x) = -\infty$

(1pt) (v) $\lim_{x \to 0^+} f(x) = \infty$

(2pts) (vi) $\lim_{x \to \infty} f(x) = 1$

(1pt) (vii) f has a removable discontinuity at $x = 4$.

other graphs are possible



6. (a) **[10 points]** Find the derivative of the function $f(x) = \frac{1}{x-1}$.

(You must use limits only)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad (2pts)$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h-1} - \frac{1}{x-1}}{h} \quad (2pts)$$

$$= \lim_{h \to 0} \frac{(x-1) - (x+h-1)}{h(x-1)(x+h-1)} \quad (2pts)$$

$$= \lim_{h \to 0} \frac{\cancel{x} \cancel{-1} \cancel{-x} - h \cancel{+1}}{h(x-1)(x+h-1)} \quad (2pts)$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}(x-1)(x+h-1)} = \lim_{h \to 0} \frac{-1}{(x-1)(x+h-1)} = \frac{-1}{(x-1)^2} \quad (2pts)$$

(b) **[6 points]** Use part (a) to find an equation of the tangent line to the curve $y = f(x)$ at the point $(2, 1)$.

$m_t = f'(2)$ (2pts). $m_t = -1$ (1pt)

$\therefore$ The equation of the tangent line is

$$y - y_1 = m_t (x - x_1)$$

$$y - 1 = -(x - 2) \quad (3pts)$$

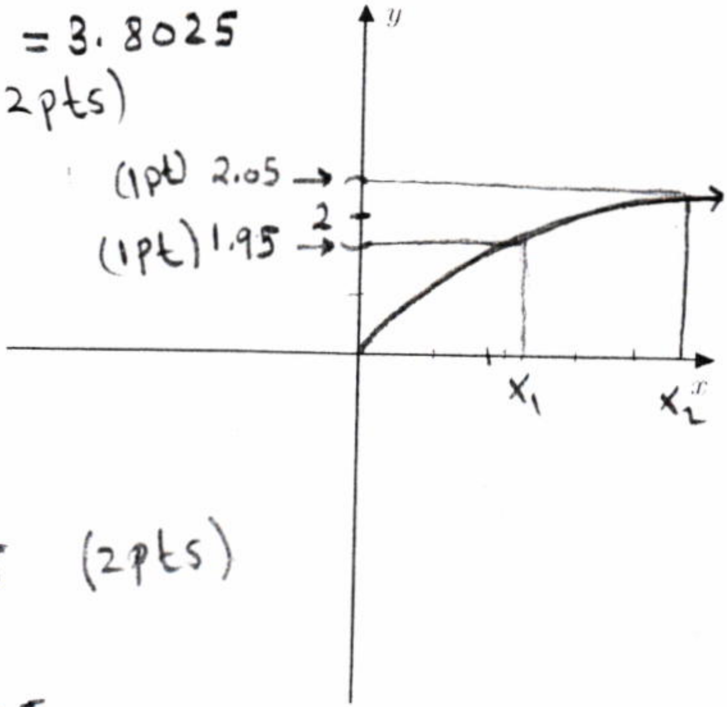
or

$$y = -x + 3$$

5. [10 points] Use the graph of $y = \sqrt{x}$ to find a number δ such that

if $|x - 4| < \delta$ then $|\sqrt{x} - 2| < 0.05$

$\sqrt{x_1} = 1.95 \Rightarrow x_1 = (1.95)^2$
$= 3.8025$
(2pts)



$\sqrt{x_2} = 2.05$

$\Rightarrow x_2 = 4.2025$ (2pts)

$\delta_1 = 4 - 3.8025$
$= 0.1975$ (1pt)

$\delta_2 = 4.2025 - 4$
$= 0.2025$ (1pt)

$\Rightarrow \delta = \min\{\delta_1, \delta_2\}$
$= 0.1975$ (2pts)