Formula for AS381 Actuarial Contingencies I

Chap 1 Economics of Insurance

1.3 *Utility theory*: when does potential customer prefer insurance policy?

indifference	**prefers** insurance policy	**avoids** insurance policy
$if\ u(w-G)=E[u(w-X)]$	$if\ u(w-G)>E[u(w-X)]$	$if\ u(w-G)<E[u(w-X)]$

Pure premium: $E[X]=\mu=$Premium   *Loaded premium*: $H=(1+a)\mu+c \quad a>0,\ c>0$

1.5 *Optimal Insurance*: Theorem 1.5.1

If a decision maker (1) has wealth amount w, (2) is *risk averse* [i.e. $u(w)$ such that $u"(w)<0$], (3) faces a random loss X, (4) will spend P on insurance, and (5) the insurance market offers all *feasible contract* of the form $I(x)$, $0\le I(x)\le x$, with $E[I(X)]=\beta$ for a price of P, *then* the decision maker's **expected utility** will be **maximized** by purchasing an insurance policy $I(x)=\begin{cases}0 & x<d^*\\ x-d^* & x\ge d^*\end{cases}$

where d^* is the solution of $\beta-\int_d^\infty(x-d)f(x)dx=0$ (or equivalently $\beta-\int_d^\infty[1-F(x)]dx=0$).

Chap 2 Individual Risk Models for a Short Term

2.2 *Models* for claim Random Variables

$X=IB$ (2.2.6) $E[X]=E[E(X|I)]$ (2.2.12) $Var(X)=Var(E[X|I])+E[Var(X|I)]$ (2.2.13)

2.3 *Sum* of Independent Claims Random Variable $F_S(s)=\Pr(S\le s)=\Pr(X+Y\le s)$ (2.3.1)

a) **Discrete** *Convolution*	b) **Continuous** *Convolution*
$F_S(s)=\sum_{all\ y\le s}\Pr(X\le s-y\|Y=y)\Pr(Y=y)$ (2.3.2)	$F_S(s)=\int_0^s\Pr(X\le s-y\|Y=y)f_Y(y)dy$ (2.3.5)
$F_S(s)=\sum_{all\ y\le s}F_X(s-y)f_Y(y)$ (2.3.3)	$F_S(s)=\int_0^s F_X(s-y)f_Y(y)dy$ (2.3.6)
$f_S(s)=\sum_{all\ y\le s}f_X(s-y)f_Y(y)$ (2.3.4)	$f_S(s)=\int_0^s f_X(s-y)f_Y(y)dy$ (2.3.7)

c) n^{th} convolution, $F^{(n)}$: *Iterative formula:* $F^{(2)}=F_2*F^{(1)}=F_2*F_1$, $F^{(3)}=F_3*F^{(2)}$, $\cdots$ $F^{(n)}=F_n*F^{(n-1)}$

d) Convolution method- *Moment Generating Function (mgf)*

$M_S(t)=E[e^{tS}]=E[e^{t(X_1+X_2+\cdots+X_n)}]=E[e^{tX_1}e^{tX_2}\cdots e^{tX_n}]$ (2.3.8)

for Independent X_i: $M_S(t)=E[e^{tX_1}]E[e^{tX_2}]\cdot\cdots\cdot E[e^{tX_n}]=M_{X_1}(t)M_{X_2}(t)M_{X_n}(t)$ (2.3.9)

2.5 *Application* (of Approximations of the Distribution of Sum) to Insurance

If $S=X_1+X_2+\cdots+X_n$ where X_k are *independent* r.v. then

$E[S]=\sum_{k=1}^n E[X_k]$ and $Var(S)=\sum_{k=1}^n Var(X_k)$ and the **Central Limit Theorem (CLT)** can be applied.

Stop-loss insurance claim under CLT:

$$E[I_d(X)]=\frac{1}{\sqrt{2\pi}\sigma}\int_d^\infty(x-d)\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]dx=\sigma\left\{\frac{\exp(-\beta^2/2)}{(2\pi)^{1/2}}-\beta\left[1-\Phi(\beta)\right]\right\} \qquad (2.5.1 \text{ and } 2.5.2)$$

Chapter 3 (a) Survival Distributions

3.2.1 *Relations* between $f(x)$, $F(x)$ and $S(x)$: $S(x)=1-F(x)$

$f(x)=\frac{d}{dx}F(x)=-\frac{d}{dx}S(x)$ (1.1MLC) $S(x)=\Pr(X>x)=\int_x^\infty f_x(t)dt=1-\int_0^x f(t)dt$ (1.2MLC or 3.2.2)

$\Pr(a<X\le b)=\int_a^b f_x(t)dt=F_x(b)-F_x(a)=S(a)-S(b)$ (1.3MLC)

3.2 *Future* Lifetime Random Variable $T(x)$

Survival Function for the Future Lifetime Random Variable $_tp_x=\frac{_{x+t}p_0}{_xp_0}=\frac{S(x+t)}{S(x)}$ (3.2.8)

3.2.2 Actuarial *Notation:* Relations between $_tp_x$, $_tq_x$ and $_{t|u}q_x$

$_tq_x=1-{}_tp_x=1-\frac{S(x+t)}{S(x)}$ (1.5MLC or 3.2.9) $_{t+u}p_x={}_tp_x\times{}_up_{x+t}$ (1.6MLC or 3.4.2)

$_{t|u}q_x={}_{t+u}q_x-{}_tq_x={}_tp_x-{}_{t+u}p_x={}_tp_x\times{}_uq_{x+t}=\frac{S(x+t)-S(x+t+u)}{S(x)}$ (1.7MLC or 3.2.10 &3.2.7)

3.2.3 *Curtate Future* Lifetime Random Variable $K(x)$

Probability Mass Function for $K(x)$: $Pr(K(x)=k)={}_kp_x\times q_{x+k}={}_{k|1}q_x={}_{k|}q_x$ $k=0,1,2,\cdots$ (1.8MLC or 3.2.11)

3.2.4 *Force of Mortality* (failure/hazard rate) $\mu(x+t)=\frac{f(x+t)}{1-F(x+t)}=-\frac{S'(x+t)}{S(x+t)}$ (1.10MLC or 3.2.13)

Relationships: $_tp_x=\frac{S(x+t)}{S(x)}=\exp\left(-\int_0^t\mu(x+u)du\right)$ (3.2.15) $f_x(x+t)=\frac{S(x+t)}{S(x)}\mu(x+t)={}_tp_x\ \mu(x+t)$ (3.2.18)

Chapter 3 (b) LifeTable: l_0 = total number from a group of newborns

3.3.*Life Table* Function: $l_x=l_0S(x)=l_0-\sum_{y=0}^{x-1}d_y=l_{x-1}p_{x-1}=l_0\left(\prod_{y=0}^{x-1}p_y\right)$ (3.3.1 , 3.4.1, and 3.4.2)

$_tp_x=\frac{S(x+t)}{S(x)}=\frac{l_{x+t}}{l_x}$ (2.1MLC) $_tq_x=\frac{S(x)-S(x+t)}{S(x)}=\frac{_td_x}{l_x}=\frac{l_x-l_{x+t}}{l_x}=1-\frac{l_{x+t}}{l_x}$ (2.3MLC)

${}_td_x = \int_0^{x+t} l_y \mu(y)dy = l_x - l_{x+t}$ (2.2MLC or 3.3.2, and 3.3.7) $\mu(x) = -\frac{S'(x)}{S(x)} = -\frac{1}{l_x}\frac{dl_x}{dx}$

3.5 *Moments* of future Lifetime Random Variables

Moments	Continuous (of $T(x)$)	Discrete of $K(x)$
1st	$\mathring{e}_x = E[T(x)] = \int_0^\infty {}_tp_x \; dt$ (2.6MLC or 3.5.2)	$e_x = \sum_{k=1}^{\infty} {}_kp_x.$ (2.9MLC or 3.5.7)
2nd	$E[T(x)^2] = 2\int_0^\infty t \cdot {}_tp_x \; dt$ (2.7MLC or 3.5.3)	$E[K(x)^2] = \sum_{k=1}^{\infty} (2k-1)_kp_x.$ (2.10MLC or 3.5.9)
temporary	$\mathring{e}_{x:\overline{n}\rceil} = \int_0^n {}_tp_x \; dt$ (2.8MLC)	$e_{x:\overline{n}\rceil} = \sum_{k=1}^{n} {}_kp_x$ (2.11MLC)
Recursion	$\mathring{e}_x = \int_0^1 {}_tp_x \; dt + p_x \mathring{e}_{x+1}$	$e_x = p_x(1+e_{x+1}).$ (2.12MLC)

3.6 Key Equation for *Fractional Age* **Assumptions** (for $0 \le r < 1, \; 0 < y < 1, \; 0 < y < 1-r$)

Assumption	${}_rq_x$	${}_{1-r}q_{x+r}$	${}_yq_{x+r}$	$\mu(x+r)$	${}_rp_x$	${}_rp_x\mu(x+r)$
UDD	rq_x	$\frac{(1-r)\,q_x}{1-rq_x}$	$\frac{yq_x}{1-rq_x}$	$\frac{q_x}{1-rq_x}$	$1-rq_x$	q_x
Constant Force	$1-(p_x)^r$	$1-(p_x)^{1-r}$	$1-(p_x)^y$	$-\ln p_x$	$(p_x)^r$	$-(p_x)^r \ln p_x$
Hyperbolic	$\frac{rq_x}{1-(1-r)\,q_x}$	$(1-r)\,q_x$	$\frac{yq_x}{1-(1-y-r)\,q_x}$	$\frac{q_x}{1-(1-r)\,q_x}$	$\frac{p_x}{1-(1-r)\,q_x}$	$\frac{q_xp_x}{[1-(1-r)\,q_x]^2}$

For UDD: $\mathring{e}_x = e_x + \frac{1}{2}$ (2.13MLC)

3.7 *Analytical* Laws (simpler alternative to Life Table)

Originator	$\mu(x)$	$S(x)$	${}_tp_x$	**Restrictions**
DeMoivre (1729)	$\frac{1}{\omega - x}$	$1-\frac{x}{\omega}$	$1-\frac{t}{\omega - x}$	$0 \le x < \omega$
Gompertz (1825)	Bc^x	$\exp[-\frac{B}{\ln c}(c^x-1)]$	$\exp\left[-\frac{B}{\ln c}c^x(c^t-1)\right]$	$B>0, \; c>1, \; x \ge 0$
Makeham (1860)	$A+Bc^x$	$\exp[-Ax-\frac{B}{\ln c}(c^x-1)]$	$\exp\left[-At-\frac{B}{\ln c}c^x(c^t-1)\right]$	like Gompertz & $A \ge -B$
Weibull (1939)	kc^x	$\exp[-\frac{k}{n+1}x^{n+1}]$	$\exp\left[\frac{k\left(x^{n+1}-(x+t)^{n+1}\right)}{n+1}\right]$	$k>0, \; n>1, \; x \ge 0$

3.8 *Select and Ultimate* Life Tables: $q_{[x]+t} \begin{cases} < q_{x+t} & t < r, \\ = q_{x+t} & t \ge r \end{cases}$ ${}_tp_{[x]+j} = \frac{l_{[x]+j+t}}{l_{[x]+j}}$

Chapter 4 Life Insurance

Present value (PV) function $Z_t = b_tv_t$ (4.2.1) Present value random variable $Z_T = b_Tv_T$ (4.2.2)

4.2.4 *Non-level (Varying)Benefit* Insurances (Continuous or payable at the moment of death)

Policy	**APV Formula for** $E(Z)$
Continuously increasing whole life	$(\bar{I}\bar{A})_x = \int_0^\infty tv^t \; {}_tp_x\mu(x+t)dt$
Annually increasing whole life	$(I\bar{A})_x = \int_0^\infty \lfloor t+1 \rfloor \; v^t \; {}_tp_x\mu(x+t)dt$
Continuously increasing n-year term	$(\bar{I}\bar{A})_{\overset{1}{x}:\overline{n}\rceil} = \int_0^n tv^t \; {}_tp_x\mu(x+t)dt$
Annually increasing n-year term	$(I\bar{A})_{\overset{1}{x}:\overline{n}\rceil} = \int_0^n \lfloor t+1 \rfloor \; v^t \; {}_tp_x\mu(x+t)dt$
Continuously increasing n-year endowment	$(\bar{I}\bar{A})_{x:\overline{n}\rceil} = (\bar{I}\bar{A})_{\overset{1}{x}:\overline{n}\rceil} + nA_{x:\overset{1}{\overline{n}\rceil}}$
Annually increasing n-year endowment	$(I\bar{A})_{x:\overline{n}\rceil} = (I\bar{A})_{\overset{1}{x}:\overline{n}\rceil} + nA_{x:\overset{1}{\overline{n}\rceil}}$
Continuously decreasing n-year term	$(\bar{D}\bar{A})_{\overset{1}{x}:\overline{n}\rceil} = \int_0^n (n-t)v^t \; {}_tp_x\mu(x+t)dt$
Annually decreasing n-year term	$(D\bar{A})_{\overset{1}{x}:\overline{n}\rceil} = \int_0^n (n-\lfloor t \rfloor)v^t \; {}_tp_x\mu(x+t)dt$

4.2.1 *Level Benefit* Insurances (Continuous or payable at the moment of death)

Policy	**Actuarial PV (APV) Formula for** $E(Z)$	$E(Z^2)$	$Var(Z)$
Whole life	$\bar{A}_x = \int_0^\infty v^t \; {}_tp_x\mu(x+t)dt$	${}^2\bar{A}_x$	${}^2\bar{A}_x - \left(\bar{A}_x\right)^2$
n-year term life	$\bar{A}_{\overset{1}{x}:\overline{n}\rceil} = \int_0^n v^t \; {}_tp_x\mu(x+t)dt$ (4.2.3)	${}^2\bar{A}_{\overset{1}{x}:\overline{n}\rceil}$	${}^2\bar{A}_{\overset{1}{x}:\overline{n}\rceil} - \left(\bar{A}_{\overset{1}{x}:\overline{n}\rceil}\right)^2$
n-year pure endowment	$A_{x:\overset{1}{\overline{n}\rceil}} = v^n \, {}_np_x$ (4.2.9)	${}^2A_{x:\overset{1}{\overline{n}\rceil}}$	$v^{2n} \, {}_np_x \times \, {}_nq_x$
n-year endowment	$\bar{A}_{x:\overline{n}\rceil} = \bar{A}_{\overset{1}{x}:\overline{n}\rceil} + A_{x:\overset{1}{\overline{n}\rceil}}$	${}^2\bar{A}_{x:\overline{n}\rceil}$	${}^2\bar{A}_{x:\overline{n}\rceil} - \left(\bar{A}_{x:\overline{n}\rceil}\right)^2$
n-year deferred whole life	${}_{n\vert}\bar{A}_x = \int_n^\infty v^t {}_tp_x\mu(x+t)dt$	${}^2_{n\vert}\bar{A}_x$	${}^2_{n\vert}\bar{A}_x - \left({}_{n\vert}\bar{A}_x\right)^2$

4.3 Insurances Payable at the End of the Year of Death (*Discrete*) Insurances

Policy	APV Formula for $E(Z)$	$E(Z^2)$	$Var(Z)$
Whole life	$A_x = \sum_{k=0}^{\infty} v^{k+1} \ {}_kp_x \ q_{x+k}$	${}^2A_x$	${}^2A_x - (A_x)^2$
n-year term life	$A_{\overset{1}{x}:\overline{n}\rceil} = \sum_{k=0}^{n-1} v^{k+1} \ {}_kp_x \ q_{x+k}$	${}^2A_{\overset{1}{x}:\overline{n}\rceil}$	${}^2A_{\overset{1}{x}:\overline{n}\rceil} - \left(A_{\overset{1}{x}:\overline{n}\rceil}\right)^2$
n-year endowment	$A_{x:\overline{n}\rceil} = A_{\overset{1}{x}:\overline{n}\rceil} + A_{x:\overset{1}{\overline{n}\rceil}}$	${}^2A_{x:\overline{n}\rceil}$	${}^2A_{x:\overline{n}\rceil} - (A_{x:\overline{n}\rceil})^2$
n-year deferred whole life	${}_{n\mid}A_x = \sum_{k=n}^{\infty} v^{k+1} \ {}_kp_x \ q_{x+k}$	${}^2_{n\mid}A_x$	${}^2_{n\mid}A_x - \left({}_{n\mid}A_x\right)^2$
Annually increasing whole life	$(IA)_x = \sum_{k=0}^{\infty} (k+1)v^{k+1} \ {}_kp_x \ q_{x+k}$	$E(Z^2)$	Use 1st principles
Annually increasing n-year term life	$(IA)_{\overset{1}{x}:\overline{n}\rceil} = \sum_{k=0}^{n-1} (k+1)v^{k+1} \ {}_kp_x \ q_{x+k}$	$E(Z^2)$	Use 1st principles
Annually increasing n-year endowment	$(IA)_{x:\overline{n}\rceil} = (IA)_{\overset{1}{x}:\overline{n}\rceil} + nA_{x:\overset{1}{\overline{n}\rceil}}$	$E(Z^2)$	Use 1st principles
Annually decreasing n-year term life	$(DA)_{\overset{1}{x}:\overline{n}\rceil} = \sum_{k=0}^{n-1} (n-k)v^{k+1} \ {}_kp_x \ q_{x+k}$	$E(Z^2)$	Use 1st principles

4.4 Relations Between Level-Benefit Policies

	Benefits Payment Structure		
Level Benefit Policy	**Continuous**	**Discrete**	**$m-$thly**
E1) n-year endowment	$\overline{A}_{x:\overline{n}\rceil} = \overline{A}_{\overset{1}{x}:\overline{n}\rceil} + A_{x:\overset{1}{\overline{n}\rceil}}$	$A_{x:\overline{n}\rceil} = A_{\overset{1}{x}:\overline{n}\rceil} + A_{x:\overset{1}{\overline{n}\rceil}}$	$A^{(m)}_{x:\overline{n}\rceil} = A^{(m)}_{\overset{1}{x}:\overline{n}\rceil} + A_{x:\overset{1}{\overline{n}\rceil}}$
E2) n-year *term* + n-year deferred whole life	$\overline{A}_x = \overline{A}_{\overset{1}{x}:\overline{n}\rceil} + {}_{n\mid}\overline{A}_x$	$A_x = A_{\overset{1}{x}:\overline{n}\rceil} + {}_{n\mid}A_x$	$A^{(m)}_x = A^{(m)}_{\overset{1}{x}:\overline{n}\rceil} + {}_{n\mid}A^{(m)}_x$
E3) n-year deferred whole life= present value of whole life purchased at $x+n$	${}_{n\mid}\overline{A}_x = v^n \ {}_np_x \ \overline{A}_{x+n}$	${}_{n\mid}A_x = v^n \ {}_np_x \ A_{x+n}$	${}_{n\mid}A^{(m)}_x = v^n \ {}_np_x \ A^{(m)}_{x+n}$

Relating $m$thly and *discrete* (annual) insurances

Policy	UDD	Accelerated Claims
Level-benefit whole life	$A^{(m)}_x = \frac{i}{i^{(m)}} A_x$	$A^{(m)}_x = (1+i)^{\frac{m-1}{2m}} A_x$
Level-benefit n-year term life	$A^{(m)}_{\overset{1}{x}:\overline{n}\rceil} = \frac{i}{i^{(m)}} A_{\overset{1}{x}:\overline{n}\rceil}$	$A^{(m)}_{\overset{1}{x}:\overline{n}\rceil} = (1+i)^{\frac{m-1}{2m}} A_{\overset{1}{x}:\overline{n}\rceil}$
Annually increasing whole life	$(IA^{(m)})_x = \frac{i}{i^{(m)}}(IA)_x$	$(IA^{(m)})_x = (1+i)^{\frac{m-1}{2m}}(IA)_x$
Annually increasing n-year term life	$(IA^{(m)})_{\overset{1}{x}:\overline{n}\rceil} = \frac{i}{i^{(m)}}(IA)_{\overset{1}{x}:\overline{n}\rceil}$	$(IA^{(m)})_{\overset{1}{x}:\overline{n}\rceil} = (1+i)^{\frac{m-1}{2m}}(IA)_{\overset{1}{x}:\overline{n}\rceil}$

Recursions

Policy	Recursion	Starting values
Level-benefit whole life	$A_x = vq_x + vp_xA_{x+1}$	$A_\omega = 0.$
Level-benefit n-year term life	$A_{\overset{1}{x}:\overline{n}\rceil} = vq_x + vp_xA_{\overset{1}{x+1}:\overline{n-1}\rceil}$	$A_{\overset{1}{y}:\overline{0}\rceil} = 0.$
Level-benefit n-year endowment	$A_{x:\overline{n}\rceil} = vq_x + vp_xA_{x+1:\overline{n-1}\rceil}$	$A_{y:\overline{0}\rceil} = 1.$
Annually increasing whole life	$(IA)_x = vq_x + vp_x\left[(IA)_{x+1} + A_{x+1}\right]$	$(IA)_\omega = 0.$
Annually increasing n-year term life	$(IA)_{\overset{1}{x}:\overline{n}\rceil} = vq_x + vp_x\left((IA)_{\overset{1}{x+1}:\overline{n-1}\rceil} + A_{\overset{1}{x+1}:\overline{n-1}\rceil}\right)$	$(IA)_{\overset{1}{y}:\overline{0}\rceil} = 0.$
Annually decreasing n-year term life	$(DA)_{\overset{1}{x}:\overline{n}\rceil} = vnq_x + vp_x(DA)_{\overset{1}{x+1}:\overline{n-1}\rceil}$	$(DA)_{\overset{1}{y}:\overline{0}\rceil}$

4.4 Relating continuous and discrete (annual) insurances Note: $\delta = \ln(1+i)$

Policy	UDD	Accelerated Claims	2nd Moments (UDD)
Level-benefit whole life	$\overline{A}_x = \frac{i}{\delta}A_x$	$\overline{A}_x = (1+i)^{1/2}A_x$	${}^2\overline{A}_x = \frac{i(2+i)}{2\delta} \ {}^2A_x$
Level-benefit n-year term life	$\overline{A}_{\overset{1}{x}:\overline{n}\rceil} = \frac{i}{\delta}A_{\overset{1}{x}:\overline{n}\rceil}$	$\overline{A}_{\overset{1}{x}:\overline{n}\rceil} = (1+i)^{1/2}A_{\overset{1}{x}:\overline{n}\rceil}$	${}^2\overline{A}_{\overset{1}{x}:\overline{n}\rceil} = \frac{i(2+i)}{2\delta} \ {}^2A_{\overset{1}{x}:\overline{n}\rceil}$
Annually increasing whole life	$(I\overline{A})_x = \frac{i}{\delta}(IA)_x$	$(I\overline{A})_x = (1+i)^{1/2}(IA)_x$	${}^2(I\overline{A})_x = \frac{i(2+i)}{2\delta} \ {}^2(IA)_x$
Annually increasing n-year term life	$(I\overline{A})_{\overset{1}{x}:\overline{n}\rceil} = \frac{i}{\delta}(IA)_{\overset{1}{x}:\overline{n}\rceil}$	$(I\overline{A})_{\overset{1}{x}:\overline{n}\rceil} = (1+i)^{1/2}(IA)_{\overset{1}{x}:\overline{n}\rceil}$	${}^2(I\overline{A})_{\overset{1}{x}:\overline{n}\rceil} = \frac{i(2+i)}{2\delta} \ {}^2(IA)_{\overset{1}{x}:\overline{n}\rceil}$

Chapter 5 Life Annuities (Payments while (x) survives**)**

AS201 formula: $\ddot{a}_{\overline{k+1}\rceil} = \frac{1-v^{k+1}}{d}$, $\overline{a}_{\overline{t}\rceil} = \frac{1-v^t}{\delta} = \frac{1-e^{-\delta t}}{\delta}$

Continuous Life Annuities

Policy	Formula(s) for $E(Y)$	Var(Y)
Whole life	$\overline{a}_x = \int_0^\infty v^t \ {}_tp_x dt$ or $\overline{a}_x = \dfrac{1-\overline{A}_x}{\delta}$	$\dfrac{1}{\delta^2}\left({}^2\overline{A}_x - \left(\overline{A}_x\right)^2\right)$ or $\dfrac{2}{\delta}\left(\overline{a}_x - {}^2\overline{a}_x\right) - \left(\overline{a}_x\right)^2$
n-year temporary	$\overline{a}_{x:\overline{n}\rceil} = \int_0^n v^t \ {}_tp_x dt$ or $\overline{a}_{x:\overline{n}\rceil} = \dfrac{1-\overline{A}_{x:\overline{n}\rceil}}{d}$	$\frac{1}{\delta^2}\left({}^2\overline{A}_{x:\overline{n}\rceil} - \left(\overline{A}_{x:\overline{n}\rceil}\right)^2\right)$
h-year deferred	${}_{h\mid}\overline{a}_x = \int_h^\infty v^t \ {}_tp_x dt$	Use first principles $\dfrac{2}{\delta}v^{2h} \cdot {}_h p_x \left(\overline{a}_{x+h} - {}^2\overline{a}_{x+h}\right) - \left({}_{h\mid}\overline{a}_x\right)^2$
n-year certain and life	$\overline{a}_{\overline{x:\overline{n}\rceil}} = \overline{a}_{\overline{n}\rceil} + {}_{n\mid}\overline{a}_x$	Use first principles
Continuously increasing whole life	$(\overline{I}\overline{a})_x = \int_0^\infty tv^t \ {}_tp_x dt$	Use first principles
Continuously increasing n-year temporary	$(\overline{I}\overline{a})_{x:\overline{n}\rceil} = \int_0^n tv^t \ {}_tp_x dt$	Use first principles

Discrete Life Annuities (Due)

Policy	Formula(s) for $E(Y)$	Var(Y)
Whole life	$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \ {}_kp_x$ or $\ddot{a}_x = \frac{1-A_x}{d}$	$\frac{1}{d^2}\left({}^2A_x - A_x^2\right)$
n-year temporary	$\ddot{a}_{x:\overline{n}\rceil} = \sum_{k=0}^{n-1} v^k \ {}_kp_x$ or $\ddot{a}_{x:\overline{n}\rceil} = \frac{1-A_{x:\overline{n}\rceil}}{d}$	$\frac{1}{d^2}\left({}^2A_{x:\overline{n}\rceil} - A^2_{x:\overline{n}\rceil}\right)$
h-year deferred	${}_{h\mid}\ddot{a}_x = {}_hE_x\ddot{a}_{x+h} = \sum_{k=n}^{\infty} v^k \ {}_kp_x$	Use first principles
n-year certain and life	$\ddot{a}_{\overline{x:\overline{n}\rceil}} = \ddot{a}_{\overline{n}\rceil} + {}_{n\mid}\ddot{a}_x$	Use first principles
Continuously increasing whole life	$(I\ddot{a})_x = \sum_{k=0}^{\infty}(k+1)v^k \ {}_kp_x$	Use first principles
Continuously increasing n-year temporary	$(I\ddot{a})_{x:\overline{n}\rceil} = \sum_{k=0}^{n-1}(k+1)v^k \ {}_kp_x$	Use first principles

$$\ddot{s}_{x:\overline{n}\rceil} = \frac{\ddot{a}_{x:\overline{n}\rceil}}{{}_nE_x} = \sum_{k=0}^{n-1} v^{n-k}\frac{l_{x+k}}{l_x}$$

Note: $d = iv$

Relations with Various Life Insurance Policies

Life Insurance Policy	At moment of death	End of year of Death
Whole life	$1 = \delta\overline{a}_x + \overline{A}_x$	$1 = d\ddot{a}_x + A_x$
	$1 = 2\delta\overline{a}_x + {}^2\overline{A}_x$	$A_x = v\ddot{a}_x - a_x$
Endowment	$1 = \delta\overline{a}_{x:\overline{n}\rceil} + \overline{A}_{x:\overline{n}\rceil}$	$1 = d\ddot{a}_{x:\overline{n}\rceil} + A_{x:\overline{n}\rceil}$
	$1 = 2\delta\overline{a}_{x:\overline{n}\rceil} + {}^2\overline{A}_{x:\overline{n}\rceil}$	$A_{x:\overline{n}\rceil} = v\ddot{a}_{x:\overline{n}\rceil} - a_{x:\overline{n-1}\rceil}$
n-yr term		$A^{\ 1}_{x:\overline{n}\rceil} = v\ddot{a}_{x:\overline{n}\rceil} - a_{x:\overline{n}\rceil}$

Relations between Various Life Annuities

Life Annuities	Equation 1	Equation 2
Continuous	$\overline{a}_x = \overline{a}_{x:\overline{n}\rceil} + {}_{n\mid}\overline{a}_x$	${}_{n\mid}\overline{a}_x = v^n \ {}_np_x\overline{a}_{x+n}$
Discrete (due)	$\ddot{a}_x = \ddot{a}_{x:\overline{n}\rceil} + {}_{n\mid}\ddot{a}_x$	${}_{n\mid}\ddot{a}_x = v^n \ {}_np_x\ddot{a}_{x+n}$
Discrete (immediate)	$a_x = a_{x:\overline{n}\rceil} + {}_{n\mid}a_x$	${}_{n\mid}a_x = v^n \ {}_np_x a_{x+n}$
$m$thly (due)	$\ddot{a}^{(m)}_x = \ddot{a}^{(m)}_{x:\overline{n}\rceil} + {}_{n\mid}\ddot{a}^{(m)}_x$	${}_{n\mid}\ddot{a}^{(m)}_x = v^n \ {}_np_x\ddot{a}^{(m)}_{x+m}$
$m$thly (immediate)	$a^{(m)}_x = a^{(m)}_{x:\overline{n}\rceil} + {}_{n\mid}a^{(m)}_x$	${}_{n\mid}a^{(m)}_x = v^n \ {}_np_x a^{(m)}_{x+m}$

Recursions for Life Annuities

Policy	Recursion	Starting values
Whole life	$\ddot{a}_x = 1 + vp_x\ddot{a}_{x+1}$	$\ddot{a}_\omega = 0.$
n-year temporary	$\ddot{a}_{x:\overline{n}\rceil} = 1 + vp_x\ddot{a}_{x+1:\overline{n-1}\rceil}$	$\ddot{a}_{y:\overline{0}\rceil} = 0.$

Policy	Whole life	n-year temporary	n-year deferred
Life Annuities Immediate	$a_x = \ddot{a}_x - 1$	$a_{x:\overline{n}\rceil} = \ddot{a}_{x:\overline{n}\rceil} - 1 + {}_nE_x$	${}_{n\mid}a_x = {}_{n\mid}\ddot{a}_x - {}_nE_x$

Approximating $m$thly life annuities using the UDD assumption

AS201 relationship: $(1+i) = (1-d)^{-1} = (1+i^{(m)}/m)^m = (1-d^{(m)}/m)^{-m}, \quad d = iv$

Policy	Approximation Formula	$\lim_{m\to\infty}$
Whole life	$\ddot{a}^{(m)}_x = \alpha(m)\ddot{a}_x - \beta(m)$	$\alpha(\infty) = \dfrac{id}{\delta^2}$
n-year temporary	$\ddot{a}^{(m)}_{x:\overline{n}\rceil} = \alpha(m)\ddot{a}_{x:\overline{n}\rceil} - \beta(m)\left(1 - {}_nE_x\right)$	$\ddot{a}^{(\infty)}_{x:\overline{n}\rceil} = \overline{a}_{x:\overline{n}\rceil}$
n-year deferred	${}_{n\mid}\ddot{a}^{(m)}_x = \alpha(m){}_{n\mid}\ddot{a}_x - \beta(m) \ {}_nE_x$	$\beta(\infty) = \dfrac{i-\delta}{\delta^2}$

$$\alpha(m) = s^{(m)}_{\overline{1}\rceil}\ddot{a}^{(m)}_{\overline{1}\rceil} = \frac{id}{i^{(m)}d^{(m)}}$$

$$\beta(m) = \frac{1}{d^{(m)}}\left(s^{(m)}_{\overline{1}\rceil} - 1\right) = \frac{i - i^{(m)}}{i^{(m)}d^{(m)}}$$

Approximation Formula

Policy	Bower's formula	Woolhouse's formula
Whole life	$\ddot{a}_x^{(m)} = \ddot{a}_x - \frac{m-1}{2m}$	$\ddot{a}_x^{(m)} = \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta+\mu_x)$
n-year temporary		$\ddot{a}_{x:\overline{n}\rceil}^{(m)} = \ddot{a}_{x:\overline{n}\rceil} - \frac{m-1}{2m}(1-v^n\ {}_np_x) - \frac{m^2-1}{12m^2}\left(\delta+\mu_x - v^n\ {}_np_x\left(\delta+\mu_{x+n}\right)\right).$
n-year deferred		${}_{n\vert}\ddot{a}_x^{(m)} = {}_{n\vert}\ \ddot{a}_x - \frac{m-1}{2m}v^n{}_np_x - \frac{m^2-1}{12m^2}v^n{}_np_x\left(\delta+\mu_{x+n}\right)$

Plans with **Apportionable Annuities**	Formula
Annuities-due Yearly	$\ddot{a}_x^{\{1\}} = E[\ddot{a}_{\overline{T}\rceil}] = E[\frac{\delta}{d}\bar{a}_{\overline{T}\rceil}] = \frac{\delta}{d}\bar{a}_x$
$m$thly Annuities-due (Under UDD)	$\ddot{a}_x^{\{m\}} = \ddot{a}_x^{(m)} - \frac{i}{d^{(m)}}\left(\frac{1}{\delta} - \frac{1}{i^{(m)}}\right)A_x = \frac{\delta}{d^{(m)}}\bar{a}_x$
Complete Annuities-immediate (Under UDD)	$\mathring{a}_x^{(m)} = a_x^{(m)} + \frac{i}{i^{(m)}}\left(\frac{1}{d^{(m)}} - \frac{1}{\delta}\right)A_x$

Chapter 6 Benefit Premium Calculation

Premium Determination Principles

Insurance Plan	**Principle I**	**Principle 2**	**Principle 3**
Idea	P = minimum annual premium so that $P(L(t)>0) \le \alpha$	$u(x)$ = linear $E[L]=0$	$u(x)$ = exponential *risk* averse
Formula	$P(T \le t_\alpha) = \alpha$	$P = E[b_T v^T]/E[Y]$	$u(x) = E[u(L)]$

6.2 Benefit Premium and the Equivalence Principle

Insurance Plan	**Fully Continuous**	**Fully Discrete**
Whole life	$\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_x} = \frac{\delta\bar{A}_x}{1-\bar{A}_x}$	$P(A_x) = \frac{A_x}{\ddot{a}_x} = \frac{dA_x}{1-A_x}$
$n-$ year term	$\bar{P}\left(\bar{A}^{1}_{x:\overline{n}\rceil}\right) = \frac{\bar{A}^{1}_{x:\overline{n}\rceil}}{\bar{a}_{x:\overline{n}\rceil}}$	$P\left(A^{1}_{x:\overline{n}\rceil}\right) = \frac{A^{1}_{x:\overline{n}\rceil}}{\ddot{a}_{x:\overline{n}\rceil}}$
$n-$ year pure endowment	$\bar{P}\left(\bar{A}_{x:\overline{n}\rceil}^{\ \ 1}\right) = \frac{\bar{A}_{x:\overline{n}\rceil}^{\ \ 1}}{\bar{a}_{x:\overline{n}\rceil}}$	$P\left(A_{x:\overline{n}\rceil}^{\ \ 1}\right) = \frac{A_{x:\overline{n}\rceil}^{\ \ 1}}{\ddot{a}_{x:\overline{n}\rceil}}$
$n-$ year endowment	$\bar{P}\left(\bar{A}_{x:\overline{n}\rceil}\right) = \frac{\bar{A}_{x:\overline{n}\rceil}}{\bar{a}_{x:\overline{n}\rceil}} = \frac{\delta\bar{A}_{x:\overline{n}\rceil}}{1-\bar{A}_{x:\overline{n}\rceil}}$	$P\left(A_{x:\overline{n}\rceil}\right) = \frac{A_{x:\overline{n}\rceil}}{\ddot{a}_{x:\overline{n}\rceil}} = \frac{dA_{x:\overline{n}\rceil}}{1-A_{x:\overline{n}\rceil}}$
$m-$ payment whole life	$\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_{x:\overline{m}\rceil}}$	$P(A_x) = \frac{A_x}{\ddot{a}_{x:\overline{m}\rceil}}$
$m-$ payment $n-$ year term	${}_m\bar{P}\left(\bar{A}^{1}_{x:\overline{n}\rceil}\right) = \frac{\bar{A}^{1}_{x:\overline{n}\rceil}}{\bar{a}_{x:\overline{m}\rceil}}$	${}_mP\left(A^{1}_{x:\overline{n}\rceil}\right) = \frac{A^{1}_{x:\overline{n}\rceil}}{\ddot{a}_{x:\overline{m}\rceil}}$
m-payment n-year endowment	${}_m\bar{P}\left(\bar{A}_{x:\overline{n}\rceil}^{\ \ 1}\right) = \frac{\bar{A}_{x:\overline{n}\rceil}^{\ \ 1}}{\bar{a}_{x:\overline{m}\rceil}}$	${}_mP\left(A_{x:\overline{n}\rceil}^{\ \ 1}\right) = \frac{A_{x:\overline{n}\rceil}^{\ \ 1}}{\ddot{a}_{x:\overline{m}\rceil}}$
m-year deferred whole life annuity	$\bar{P}\left({}_{m\vert}\bar{a}_x\right) = \frac{\bar{A}_{x:\overline{m}\rceil}^{\ \ 1}\bar{a}_{x+m}}{\bar{a}_{x:\overline{m}\rceil}}$	$P\left({}_{m\vert}\ddot{a}_x\right) = \frac{A_{x:\overline{m}\rceil}^{\ \ 1}\ddot{a}_{x+m}}{\ddot{a}_{x:\overline{m}\rceil}}$

	Fully Continuous Benefit Premium	Fully Discrete Benefit Premium
n-year Endowment of 1	$\bar{P}\left(\bar{A}_{x:\overline{n}\rceil}\right) = \frac{\bar{A}_{x:\overline{n}\rceil}}{\bar{a}_{x:\overline{n}\rceil}} = \frac{\delta\bar{A}_{x:\overline{n}\rceil}}{1-\bar{A}_{x:\overline{n}\rceil}} = \frac{1}{\bar{a}_{x:\overline{n}\rceil}} - \delta$	$P_{x:\overline{n}\rceil} = \frac{A_{x:\overline{n}\rceil}}{\ddot{a}_{x:\overline{n}\rceil}} = \frac{dA_{x:\overline{n}\rceil}}{1-A_{x:\overline{n}\rceil}} = \frac{1}{\ddot{a}_{x:\overline{n}\rceil}} - d$

Relation between Term, Pure Endowment and Endowment Insurance ${}_hP_{x:\overline{n}\rceil} = {}_hP^{1}_{x:\overline{n}\rceil} + {}_hP_{x:\overline{n}\rceil}^{\ \ 1}$

Plan	**Var**$({}_0L)$ where S=sum insured (i.e., $b_T = S$)
Fully continuous whole life	$\left(S+\frac{\pi}{\delta}\right)^2\left({}^2\overline{A}_x-\overline{A}_x^2\right)$
Fully continuous n-year endowment	$\left(S+\frac{\pi}{\delta}\right)^2\left[{}^2\overline{A}_{x:\overline{n}\rceil}-\left(\overline{A}_{x:\overline{n}\rceil}\right)^2\right]$
Fully continuous n-year term	$S^2\left({}^2\overline{A}^1_{x:\overline{n}\rceil}-\left(\overline{A}^1_{x:\overline{n}\rceil}\right)^2\right)+\left(\frac{\pi}{\delta}\right)^2\left[{}^2\overline{A}_{x:\overline{n}\rceil}-\left(\overline{A}_{x:\overline{n}\rceil}\right)^2\right]+\frac{2S\pi}{\delta}\left({}^2\overline{A}^1_{x:\overline{n}\rceil}-\overline{A}^1_{x:\overline{n}\rceil}\overline{A}_{x:\overline{n}\rceil}\right)$
Fully continuous n-year pure endowment	$S^2\left({}^2\overline{A}_{x:\overset{1}{\overline{n}\rceil}}-\left(\overline{A}_{x:\overset{1}{\overline{n}\rceil}}\right)^2\right)+\left(\frac{\pi}{\delta}\right)^2\left[{}^2\overline{A}_{x:\overline{n}\rceil}-\left(\overline{A}_{x:\overline{n}\rceil}\right)^2\right]+\frac{2S\pi}{\delta}\left({}^2\overline{A}_{x:\overset{1}{\overline{n}\rceil}}-\overline{A}_{x:\overset{1}{\overline{n}\rceil}}\overline{A}_{x:\overline{n}\rceil}\right)$
Fully discrete whole life	$\left(S+\frac{\pi}{d}\right)^2\left({}^2A_x-A_x^2\right)$
Fully discrete n-year endowment	$\left(S+\frac{\pi}{d}\right)^2\left[{}^2\overline{A}_{x:\overline{n}\rceil}-\left(\overline{A}_{x:\overline{n}\rceil}\right)^2\right]$
Fully discrete n-year term	$S^2\left({}^2\overline{A}^1_{x:\overline{n}\rceil}-\left(\overline{A}^1_{x:\overline{n}\rceil}\right)^2\right)+\left(\frac{\pi}{\delta}\right)^2\left[{}^2\overline{A}_{x:\overline{n}\rceil}-\left(\overline{A}_{x:\overline{n}\rceil}\right)^2\right]+\frac{2S\pi}{d}\left({}^2\overline{A}^1_{x:\overline{n}\rceil}-\overline{A}^1_{x:\overline{n}\rceil}\overline{A}_{x:\overline{n}\rceil}\right)$
Fully discrete n-year pure endowment	$S^2\left({}^2\overline{A}_{x:\overset{1}{\overline{n}\rceil}}-\left(\overline{A}_{x:\overset{1}{\overline{n}\rceil}}\right)^2\right)+\left(\frac{\pi}{d}\right)^2\left[{}^2\overline{A}_{x:\overline{n}\rceil}-\left(\overline{A}_{x:\overline{n}\rceil}\right)^2\right]+\frac{2S\pi}{d}\left({}^2\overline{A}_{x:\overset{1}{\overline{n}\rceil}}-\overline{A}_{x:\overset{1}{\overline{n}\rceil}}\overline{A}_{x:\overline{n}\rceil}\right)$

Percentile Premium for Insurance Payable at:
a) end of year (Fully Discrete): Find integer r such that ${}_rq_x \le \alpha < {}_{r+1}q_x$
b) the **Moment** of Death: Smallest premium so that $Pr({}_0L > 0) \le \alpha \iff Pr(T \le t_\alpha) = \alpha \rightarrow {}_{t_\alpha}q_x = \alpha$.

where S=sum insured (i.e., $b_T = S$)	**Discrete**		**Continuous**	
Type of Plan	${}_nq_x \le \alpha$	${}_nq_x > \alpha$	${}_nq_x \le \alpha$	${}_nq_x > \alpha$
Whole life	$\frac{S}{\ddot{s}_{\overline{r+1}\rceil}}$	$\frac{S}{\ddot{s}_{\overline{r+1}\rceil}}$	$\frac{S}{\bar{s}_{\overline{t_\alpha}\rceil}}$	$\frac{S}{\bar{s}_{\overline{t_\alpha}\rceil}}$
n-year term	0	$\frac{S}{\ddot{s}_{\overline{r+1}\rceil}}$	0	$\frac{S}{\bar{s}_{\overline{t_\alpha}\rceil}}$
n-year endowment	$\frac{S}{\ddot{s}_{\overline{n}\rceil}}$	$\frac{S}{\ddot{s}_{\overline{r+1}\rceil}}$	$\frac{S}{\bar{s}_{\overline{n}\rceil}}$	$\frac{S}{\bar{s}_{\overline{t_\alpha}\rceil}}$

Apportionable Premiums [policies with Premium Refunds(PR)]

m**thly apportionable premium**	**yearly apportionable premium**
${}_hP^{\{m\}}\left(\overline{A}_{x:\overline{n}\rceil}\right)=\frac{\overline{A}_{x:\overline{n}\rceil}}{\ddot{a}^{\{m\}}_{x:\overline{h}\rceil}}=\frac{d^{(m)}\overline{A}_{x:\overline{n}\rceil}}{\delta\bar{a}_{x:\overline{h}\rceil}}=\frac{d^{(m)}}{\delta}\ {}_h\bar{P}\left(\overline{A}_{x:\overline{n}\rceil}\right)$	$P^{\{1\}}\left(\overline{A}_x\right)=\frac{d\overline{A}_x}{\delta\ddot{a}_x}=\frac{d}{\delta}\bar{P}\left(\overline{A}_x\right)=\bar{a}_{\overline{1}\rceil}\bar{P}\left(\overline{A}_x\right)$
Difference: $P\left(\overline{A}_x^{PR}\right)={}_h P^{\{m\}}\left(\overline{A}_x\right)-{}_h P^{(m)}\left(\overline{A}_x\right)$	$P\left(\overline{A}_x^{PR}\right)=\frac{\bar{P}\left(\overline{A}_x\right)\left(\overline{A}_x-A_x\right)}{\delta\ddot{a}_x}=\bar{P}\left(\overline{A}_x\right)\left(\frac{d}{\delta}-\frac{\bar{a}_x}{\ddot{a}_x}\right)$

Chapter 7 Benefit Reserves

$$Var[{}_tL|T(x)>t]=\left[1+\frac{\overline{P}(\overline{A}_x)}{\delta}\right]^2Var[v^{T(x)-t}|T(x)>t]=\left[1+\frac{\overline{P}(\overline{A}_x)}{\delta}\right]^2[{}^2\bar{A}_{x+t}-(\bar{A}_{x+t})^2]$$

$$f_{{}_tL}(y)=\frac{1}{(\delta y+\overline{P}\left(\overline{A}_x\right))(1-F_{T(x)}(t))}f_{T(x)}\left(t-\frac{1}{\delta}\log\left[\frac{\delta y+\overline{P}\left(\overline{A}_x\right)}{\delta+\overline{P}\left(\overline{A}_x\right)}\right]\right)=\frac{1}{(\delta y+\overline{P}\left(\overline{A}_x\right))}f_{T(x)}\left(-\frac{1}{\delta}\log\left[\frac{\delta y+\overline{P}\left(\overline{A}_x\right)}{\delta+\overline{P}\left(\overline{A}_x\right)}\right]\right)$$

$$F_{{}_tL}(y)=\Pr({}_tL|T(x)>t)=\frac{\Pr(T(x)\ge t-(1/\delta)\log\{[\delta y+\overline{P}\left(\overline{A}_x\right)]/[\delta+\overline{P}\left(\overline{A}_x\right)]\})}{\Pr(T(x)>t)}$$

$$=\frac{1-F_{T(x)}(t-(1/\delta)\{\log[\delta y+\overline{P}\left(\overline{A}_x\right)]/[\delta+\overline{P}\left(\overline{A}_x\right)]\})}{1-F_{T(x)}(t)}=1-F_{T(x+t)}\left(t-\frac{1}{\delta}\log\left[\frac{\delta y+\overline{P}\left(\overline{A}_x\right)}{\delta+\overline{P}\left(\overline{A}_x\right)}\right]\right)$$

Fully Continuous Benefit Reserves to (x) with **unit** benefit for duration t

Insurance Plan	**Prospective Formula**
n-year term Ins	${}_t\bar{V}(\bar{A}^{\;1}_{x:\overline{n}\rceil}) = \begin{cases} \bar{A}^{\;1}_{x+t:\overline{n-t}\rceil} - \bar{P}(\bar{A}^{\;1}_{x:\overline{n}\rceil})\bar{a}_{x+t:\overline{n-t}\rceil} & t<n \\ 0 & t=n \end{cases}$
n-year endowment Ins	${}_t\bar{V}(\bar{A}_{x:\overline{n}\rceil}) = \begin{cases} \bar{A}_{x+t:\overline{n-t}\rceil} - \bar{P}(\bar{A}_{x:\overline{n}\rceil})\bar{a}_{x+t:\overline{n-t}\rceil} & t<n \\ 1 & t=n \end{cases}$
h-payment whole life Ins	${}^h_t\bar{V}(\bar{A}_x) = \begin{cases} \bar{A}_{x+t} - {}_h\bar{P}(\bar{A}_x)\bar{a}_{x+t:\overline{h-t}\rceil} & t \le h \\ \bar{A}_{x+t} & t>h \end{cases}$
h-payment $n-$year endowment Ins	${}^h_t\bar{V}(\bar{A}_{x:\overline{n}\rceil}) = \begin{cases} \bar{A}_{x+t:\overline{n-t}\rceil} - {}_h\bar{P}(\bar{A}_{x:\overline{n}\rceil})\bar{a}_{x+t:\overline{h-t}\rceil} & t \le h < n \\ \bar{A}_{x+t:\overline{n-t}\rceil} & h<t<n \\ 1 & t=n \end{cases}$
n-year pure endowment Ins	${}_t\bar{V}(\bar{A}_{x:\overset{1}{\overline{n}\rceil}}) = \begin{cases} \bar{A}_{x+t:\overset{1}{\overline{n-t}\rceil}} - \bar{P}\left(\bar{A}_{x:\overset{1}{\overline{n}\rceil}}\right)\bar{a}_{x+t:\overline{n-t}\rceil} & t<n \\ 1 & t=n \end{cases}$
Whole life annuity	${}_t\bar{V}({}_{n\vert}\bar{a}_x) = \begin{cases} {}_{n-t\vert}\bar{a}_{x+t} - \bar{P}\left({}_{n\vert}\bar{a}_x\right)\bar{a}_{x+t:\overline{n-t}\rceil} & t \le n \\ \bar{a}_{x+t} & t>n \end{cases}$

Alternative Formula	$n-$ year endowment ${}_t\bar{V}(\bar{A}_{x:\overline{n}\rceil})$	whole life insurance ${}_t\bar{V}(\bar{A}_x)$
Prospective	$= \bar{A}_{x+t:\overline{n-t}\rceil} - \bar{P}(\bar{A}_{x:\overline{n}\rceil})\bar{a}_{x+t:\overline{n-t}\rceil}$	$\bar{A}_{x+t} - \bar{P}(\bar{A}_x)\bar{a}_{x+t} = 1 - \dfrac{\bar{a}_{x+t}}{\bar{a}_x} = \dfrac{\bar{A}_{x+t} - \bar{A}_x}{1-\bar{A}_x}$
Premium-difference	$= \left[\bar{P}(\bar{A}_{x+t:\overline{n-t}\rceil}) - \bar{P}(\bar{A}_{x:\overline{n}\rceil})\right]\bar{a}_{x+t:\overline{n-t}\rceil}$	$= \dfrac{\left[\bar{P}(\bar{A}_{x+t}) - \bar{P}(\bar{A}_{x:\overline{n}\rceil})\right]}{\bar{P}(\bar{A}_{x+t}) + \delta}$
Paid-up insurance	$= \left[1 - \dfrac{\bar{P}(\bar{A}_{x:\overline{n}\rceil})}{\bar{P}(\bar{A}_{x+t:\overline{n-t}\rceil})}\right]\bar{A}_{x+t:\overline{n-t}\rceil}$	$= \left[1 - \dfrac{\bar{P}(\bar{A}_x)}{\bar{P}(\bar{A}_{x+t})}\right]\bar{A}_{x+t}$
Retrospective	$= \dfrac{1}{{}_tE_x}\left[\bar{P}(\bar{A}_{x:\overline{n}\rceil})\bar{a}_{x:\overline{t}\rceil} - \bar{A}^{\;1}_{x:\overline{t}\rceil}\right]$	

Fully Discrete Benefit Reserves to (x) with **unit** benefit for duration k

Insurance Plan	**Prospective Formula**
n-year term Ins	${}_kV^{\;1}_{x:\overline{n}\rceil} = \begin{cases} A^{\;1}_{x+k:\overline{n-k}\rceil} - P^{\;1}_{x:\overline{n}\rceil}\ddot{a}_{x+k:\overline{n-k}\rceil} & k<n \\ 0 & k=n \end{cases}$
n-year endowment Ins	${}_kV_{x:\overline{n}\rceil} = \begin{cases} A_{x+k:\overline{n-k}\rceil} - P_{x:\overline{n}\rceil}\ddot{a}_{x+k:\overline{n-k}\rceil} = 1 - \dfrac{\ddot{a}_{x+k:\overline{n-k}\vert}}{\ddot{a}_{x:\overline{t}\rceil}} = \dfrac{A_{x+k:\overline{n-k}\vert} - A_{x:\overline{n}\rceil}}{1-A_{x:\overline{n}\rceil}} & k<n \\ 1 & k=n \end{cases}$
h-payment whole life Ins	${}^h_kV_x = \begin{cases} A_{x+k} - {}_hP_x\ddot{a}_{x+k:\overline{h-k}\rceil} & k<h \\ A_{x+k} & k \ge h \end{cases}$
h-payment $n-$year endowment Ins	${}^h_kV_{x:\overline{n}\rceil} = \begin{cases} A_{x+k:\overline{n-k}\rceil} - {}_hP_{x:\overline{n}\rceil}\ddot{a}_{x+k:\overline{n-k}\rceil} & k<h<n \\ \bar{A}_{x+t:\overline{n-t}\rceil} & h \le k < n \\ 1 & k=n \end{cases}$
n-year pure endowment Ins	${}_kV_{x:\overset{1}{\overline{n}\rceil}} = \begin{cases} A_{x+k:\overset{1}{\overline{n-k}\rceil}} - P_{x:\overset{1}{\overline{n}\rceil}}\ddot{a}_{x+k:\overline{n-k}\rceil} & k<n \\ 1 & k=n \end{cases}$
Whole life annuity	${}_kV({}_{n\vert}\ddot{a}_x) = \begin{cases} {}_{n-k\vert}\ddot{a}_{x+k} - P\left({}_{n\vert}\ddot{a}_x\right)\ddot{a}_{x+k:\overline{n-k}\rceil} & k<n \\ \ddot{a}_{x+k} & k \ge n \end{cases}$

$P_x = P^{\;1}_{x:\overline{n}\rceil} + P_{x:\overset{1}{\overline{n}\rceil}}\ {}_nV_x.$ $\qquad Var[{}_kL \vert K(x) > k-1] = \left[1 + \dfrac{P_{x:\overline{n}\rceil}}{d}\right]^2 [{}^2A_{x+k:\overline{n-k}\rceil} - (A_{x+k:\overline{n-k}\rceil})^2]$

Alternative Formula	$n-$ year endowment ${}_kV_{x:\overline{n}\rceil}$	whole life insurance ${}_kV_x$
Prospective	$= A_{x+k:\overline{n-k}\rceil} - P_{x:\overline{n}\rceil}\ddot{a}_{x+k:\overline{n-k}\rceil}$	$A_{x+k} - P_x\ddot{a}_{x+k} = 1 - \dfrac{\ddot{a}_{x+k}}{\ddot{a}_x} = \dfrac{A_{x+k} - A_x}{1-A_x}$
Premium-difference	$= (P_{x+k:\overline{n-k}\rceil} - P_{x:\overline{n}\rceil})\ddot{a}_{x+k:\overline{n-k}\rceil}$	$= \dfrac{[P_{x+k} - P_x]}{P_{x+k} + d}$
Paid-up insurance	$= \left[1 - \dfrac{P_{x:\overline{n}\rceil}}{P_{x+k:\overline{n-k}\rceil}}\right]A_{x+k:\overline{n-k}\rceil}$	$= \left[1 - \dfrac{P_x}{P_{x+k}}\right]A_{x+k}$
Retrospective	$= \dfrac{1}{{}_kE_x}\left[P_{x:\overline{n}\rceil}\ddot{a}_{x:\overline{k}\rceil} - \bar{A}^{\;1}_{x:\overline{k}\rceil}\right]$	

Benefit Reserves to (x) based on True m-thly Benefit Premiums (max payment is $h < n$)

Case	Prospective formula $(k < h)$
a) Fully Discrete	${}_k^hV_{x:\overline{n}\rceil}^{(m)} = A_{x+k:\overline{n-k}\rceil} - {}_hP_{x:\overline{n}\rceil}^{(m)}\ddot{a}_{x+k:\overline{h-k}\rceil}^{(m)}$ (7.6.1)
b) Semi-continuous	${}_k^hV^{(m)}\left(\bar{A}_{x:\overline{n}\rceil}\right) = \bar{A}_{x+k:\overline{n-k}\rceil} - {}_hP^{(m)}\left(\bar{A}_{x:\overline{n}\rceil}\right)\ddot{a}_{x+k:\overline{h-k}\rceil}^{(m)} = \frac{i}{\delta}\cdot{}_k^h V_{x:\overline{n}\rceil}^{(m)}$ (7.6.4)

Case	Relations to other reserves $(k < h)$
a) Fully Discrete	${}_k^hV_{x:\overline{n}\rceil}^{(m)} = {}_k^h V_{x:\overline{n}\rceil} + {}_hP_{x:\overline{n}\rceil}\ddot{a}_{x+k:\overline{h-k}\rceil} - {}_hP_{x:\overline{n}\rceil}^{(m)}\ddot{a}_{x+k:\overline{h-k}\rceil}^{(m)}$ ${}_k^hV_{x:\overline{n}\rceil}^{(m)} = {}_k^h V_{x:\overline{n}\rceil} + {}_hP_{x:\overline{n}\rceil}^{(m)}\left(\frac{\ddot{a}_{x:\overline{h}\rceil}^{(m)}}{\ddot{a}_{x:\overline{h}\rceil}}\ddot{a}_{x+k:\overline{h-k}\rceil} - \ddot{a}_{x+k:\overline{h-k}\rceil}^{(m)}\right) = {}_k^h V_{x:\overline{n}\rceil} + \beta(m)\cdot{}_hP_{x:\overline{n}\rceil}^{(m)}\cdot{}_kV_{\overset{1}{x}:\overline{h}\rceil}$
b) Semi-continuous	${}_k^hV^{(m)}\left(\bar{A}_{x:\overline{n}\rceil}\right) = {}_k^hV\left(\bar{A}_{x:\overline{n}\rceil}\right) + \beta(m)\cdot{}_hP^{(m)}\left(\bar{A}_{x:\overline{n}\rceil}\right)\cdot{}_kV_{\overset{1}{x}:\overline{h}\rceil}$
c) Fully-continuous	${}_k^h\bar{V}\left(\bar{A}_{x:\overline{n}\rceil}\right) = {}_k^hV\left(\bar{A}_{x:\overline{n}\rceil}\right) + \beta(\infty)\cdot{}_h\bar{P}\left(\bar{A}_{x:\overline{n}\rceil}\right)\cdot{}_kV_{\overset{1}{x}:\overline{h}\rceil}$

Benefit Reserves to (x) based on **Apportionable** or Discounted Continuous Basis: $k < h$

$${}_k^hV^{\{m\}}\left(\bar{A}_{x:\overline{n}\rceil}\right) = \bar{A}_{x+k:\overline{n-k}\rceil} - {}_hP^{\{m\}}\left(\bar{A}_{x:\overline{n}\rceil}\right)\ddot{a}_{x+k:\overline{h-k}\rceil}^{\{m\}} = \bar{A}_{x+k:\overline{n-k}\rceil} - {}_h\bar{P}\left(\bar{A}_{x:\overline{n}\rceil}\right)\bar{a}_{x+k:\overline{h-k}\rceil} = {}_k^h\bar{V}\left(\bar{A}_{x:\overline{n}\rceil}\right) \quad (7.7.1 \& 7.7.2)$$

$$P^{\{1\}}\left(\bar{A}_x\right) = P\left(\bar{A}_x\right) + P\left(\bar{A}_x^{PR}\right) \quad (7.7.3) \qquad {}_kV^{\{1\}}\left(\bar{A}_x\right) = {}_kV\left(\bar{A}_x\right) + {}_kV\left(\bar{A}_x^{PR}\right) \quad (7.7.4)$$

Chapter 8 Analysis of Benefit Reserves (more general than Chap 7**)**

fully discrete policies (8.2.4)	${}_kV = \sum_{j=0}^{\infty} b_{k+j+1}v^{j+1}\ {}_jp_{x+k}q_{x+k+j} - \sum_{j=0}^{\infty}\pi_{k+j}v^j\ {}_jp_{x+k}$
fully continuous (8.2.9)	${}_t\bar{V} = \int_0^{\infty} b_{t+u}v^u\ {}_up_{x+t}\mu_x(t+u)du - \int_0^{\infty}\pi_{t+r}v^r\ {}_rp_{x+t}dr$

8.2 The Recursive Approach: Basic Idea

Recursion Relation for Benefit Reserves	Recursion Formula
Backwards Recursion	$\left({}_tV + \pi_t\right)(1+i) = b_{t+1}q_{x+t} + p_{x+t\,t+1}V$
Fackler's Accumulation (forward recursion)Formula	${}_{t+1}V = \frac{\left({}_tV + \pi_t\right)(1+i) - b_{t+1}q_{x+t}}{p_{x+t}}$
in terms of the **Net Amount at Risk**	$\left({}_tV + \pi_t\right)(1+i) = {}_{t+1}V + \left(b_{t+1} - {}_{t+1}V\right)q_{x+t}$

The **Hattendorf Theorem** (for obtaining variance of conditional Loss r.v.)

$$Var\left({}_kL \mid K_x \geq k\right) = \begin{cases} \sum_{j=k}^{\infty} v^{2(j-k)}Var(\Lambda_j|K_x \geq k) \\ \sum_{j=k}^{\infty} v^{2(j-k)}\ {}_{j-k}p_{x+k}\{[v(b_{j+1} - {}_{j+1}V)]^2 p_{x+j}q_{x+j}\} \\ \sum_{j=k}^{k+r-1} v^{2(j-k)}\ {}_{j-k}p_{x+k}\{[v(b_{j+1} - {}_{j+1}V)]^2 p_{x+j}q_{x+j}\} + v^{2r}\ {}_rp_{x+k}Var\left({}_{k+r}L \mid K_x \geq k+r\right) \end{cases}$$

where $Var(\Lambda_j|K_x \geq k) = {}_{j-k}p_{x+k}Var(\Lambda_j|K_x \geq j) = {}_{j-k}p_{x+k}\left[v(b_{j+1} - {}_{j+1}V)\right]^2 p_{x+j}q_{x+j}$.

8.3 The Recursive Approach: Further Applications for fractional periods

Interim Benefit Reserves $0 < s \leq 1$ and k and $k+1$ are premium payment dates

Backward	$\left({}_kV + \pi_k\right)(1+i)^s = v^{1-s}b_{k+1\,s}q_{x+k} + {}_sp_{x+k\,k+s}V$
Forward	${}_{k+s}V(1+i)^{1-s} = b_{k+1\,1-s}q_{x+k+s} + {}_{1-s}p_{x+k+s\,k+1}V$

Interim Benefit Reserves with m**thly** paid premiums $0 < r \leq 1/m$, $h = 0, 1, ..., m-1$,
and $k + (h/m)$ and $k + (h/m) + 1$ are premium payment dates

m**thly** Policy	approximations
regular mthly	${}_{k+(h/m)+r}V^{(m)} = (1 - \frac{h}{m} - r){}_kV^{(m)} + (\frac{h}{m} + r){}_{k+1}V^{(m)} + (\frac{1}{m} - r)P^{(m)}$
apportionable premium	${}_{k+(h/m)+r}V^{\{m\}} = (1 - \frac{h}{m} - r){}_kV^{\{m\}} + (\frac{h}{m} + r){}_{k+1}V^{\{m\}} + (\frac{1}{m} - r)P^{\{m\}}$

Chapter X Insurance Models Including Expenses

X.2 Gross Premium Reserve

Thiele's Differential Equation for Gross Premium Reserve

$$\frac{d_tV^g}{dt} = G_t(1-c_t) - e_t + \left(\delta_t + \mu_{x+t}\right)_tV^g - (b_t + E_t)\mu_{x+t}$$

X.3 Expense Reserve and Full Preliminary Term (FPT) Reserve

Recursion Relation for Expense Reserves

$$\left({}_hV^e + P_h^e - R_h\right)(1+i) = q_{x+h}\ E_{h+1} + p_{x+h}\ {}_{h+1}V^e$$

X.4 Basis, Asset Share and Profit

Recursion Relation for Asset Shares

$$\left[{}_hAS + G_h(1-c_h) - e_h\right](1+i) = q_{x+h}\left(b_{h+1} + E_{h+1}\right) + p_{x+h}\ {}_{h+1}AS$$

Chapter 20 Pension Mathematics

20.1 The Salary Scale Function

Salary Scale Function

$$\frac{s_y}{s_x} = \frac{\text{salary received in year of age } y \text{ to } y+1}{\text{salary received in year of age } x \text{ to } x+1}$$