Q1.

(a) Find a matrix P that diagonalizes $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

(b) Find A^{10}.

(a) $|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 - (1-\lambda)$

$= (1-\lambda)\left((1-\lambda)^2 - 1\right)$

$= (1-\lambda)(\lambda^2 - 2\lambda) = (1-\lambda)(\lambda-2)\lambda$

2

$0, 1, 2$ are the eigenvalues.

2 $\lambda = 0$ $\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \vec{K}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $K_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

2 $\lambda = 1$ $\begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix} \vec{K}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $K_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

2 $\lambda = 2$ $\begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix} K_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $K_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

2 $P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$ $P^{-1} = \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ 1/2 & -1 & 1/2 \end{pmatrix}$ 3 by Gauss-Jordan or minor method

6 $P^{-1}AP = D = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 2 \end{pmatrix}$ $P^{-1}A^{10}P = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 2^{10} \end{pmatrix}$ 6

$A^{10} = P D^{10} P^{-1} = \begin{pmatrix} 2^9 & 1-2^{10} & 2^9 \\ 0 & 1 & 0 \\ 2^9 & 1-2^{10} & 2^9 \end{pmatrix}$ 2 1

Q2. Using divergence theorem, find the outward flux of

$$\mathbf{E} = 2xyz\,\hat{\mathbf{a}}_x + x^2 z\,\hat{\mathbf{a}}_y + x^2 y\,\hat{\mathbf{a}}_z,$$

for the region defined by $0 < x < 2,\ 1 < y < 2, -1 < z < 3$.

$\nabla \cdot E = 2yz$ 6

$\oint E \cdot ds = \int_{-1}^{3}\int_{1}^{2}\int_{0}^{2} 2yz\, dx\, dy\, dz$ 3

$= \int_{-1}^{3}\int_{1}^{2} 4yz\, dy\, dz$ 2 $= \int_{-1}^{3} 6z\, dz$ 2 $= 24$. 2

Q3.

(a) If $f(z) = 2x - x^3 + 3xy^2 + iv(x, y)$ is analytic for any z, find $v(x,y)$ and write f in terms of z.

(b) Find all points when the derivative of $f(z) = x^2 + iy^2$ exists and find $f'(z)$ whenever it exists.

(a) $u = 2x - x^3 + 3xy^2$

$u_x = 2 - 3x^2 + 3y^2 = v_y$

$v = 2y - 3yx^2 + y^3 + g(x)$ (2)

$v_x = -6yx + g'(x) = -u_y = -6xy$

$\Rightarrow g'(x) = 0$ (2)

$v = 2y - 3yx^2 + y^3$ (1)

$f = u + iv = 2(x+iy) - (x+iy)^3 = 2z - z^3$ (4)

(b) $u_x = v_y \quad 2x = 2y$ $\quad f'$ exist on the line $x = y$ (2)

$u_y = -v_x \quad 0 = 0$

(2)

$f'(z) = u_x + iv_x = 2x + \cancel{\ }$

(2)

Q4.

(a) Find all complex numbers z satisfying the equation $ln(z) = (\pi/2)i$.

(b) Solve the equation $\cos z = \sqrt{2}$.

(a) $\ln z = \log_e|z| + i(\arg z + 2\pi k) = \frac{\pi}{2} i$ 1

$\Rightarrow \log_e|z| = 0 \qquad \arg z + 2\pi k = \pi/2$

$|z| = 1$ 1 $\qquad \arg z = \pi/2 + 2\pi k$ 1

$z = i$ (2)

(b) $\dfrac{e^{iz} + e^{-iz}}{2} = \sqrt{2}$ (2). $\Rightarrow e^{2iz} - 2\sqrt{2}e^{iz} + 1$

$e^{iz} = \mp 1 + \sqrt{2}$ (2)

$iz = \ln(1+\sqrt{2})$ or $\ln(-1+\sqrt{2})$ (1)

$iz = \log_e(1+\sqrt{2}) + i\arg(1+\sqrt{2}) \Rightarrow z = -i\log_e(1+\sqrt{2}) + 2n\pi$ (1) (1.5)

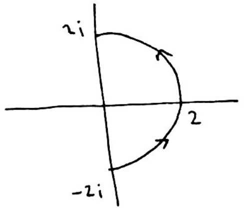
$\arg(1+\sqrt{2}) = 0 + 2n\pi$

$iz = \log_e(-1+\sqrt{2}) + i\arg(-1+\sqrt{2}) \Rightarrow z = -i\log_e(-1+\sqrt{2}) + 2n\pi$ (1) (1.5)

$\arg(-1+\sqrt{2}) = 0 + 2n\pi$

$n = 0, \mp 1, \mp 2, \ldots$

Q5. Evaluate $\int_C \bar{z}dz$ along the contour C, which is right-hand half of the circle $|z| = 2$, oriented counter-clockwise.



$x = 2\cos t \quad y = 2\sin t \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$ 4

$\Rightarrow z = 2e^{it}$ 3 $\quad \bar{z} = 2e^{-it}$

$dz = 2ie^{it}\,dt$ 3

$$\int_{-\pi/2}^{\pi/2} 4i\,dt = 4\pi i$$

3 2

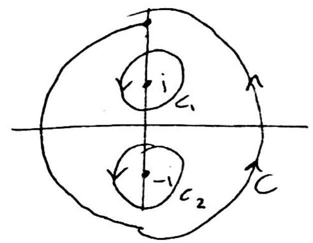
Q6. Use Cauchy's integral formula to evaluate,

$$\oint_C \left[\frac{\cos z}{(z^2+1)^2} - 2\cosh z\right] dz,$$

C is the circle $|z + i| = 3$.

$\cosh z$ is analytic in $|z+i| = 3$. Hence $\oint -2\cosh z\,dz = 0$ 3

$$I = \oint_C \frac{\cos z}{(z-i)^2(z+i)^2}\,dz$$



By deformation thm.

$$I = \oint_{C_1} + \oint_{C_2}$$ 2

$$\oint_{C_1} = 2\pi i\left(\frac{\cos z}{(z+i)^2}\right)'_{z=i} = 2\pi i\left(\frac{\sin z(z+i)^2 - 2\cos z(z+i)}{(z+i)^4}\right)_{z=i}$$

$$= 2\pi i\left(\frac{\sin i\,(2i)^2 - 2\cos i\,(2i)}{(2i)^4}\right)$$ 4

$$\oint_{C_2} = 2\pi i\left(\frac{\cos z}{(z-i)^2}\right)'_{z=-i} = 2\pi i\left(\frac{\sin(-i)(-2i)^2 - 2\cos(-i)(-2i)}{(-2i)^4}\right)$$ 4

$$\oint_{C_1} + \oint_{C_2} = 0$$ 2

Q7. Give two Laurent series for the function $f(z) = \frac{2z}{(z-1)(z-3)}$ in the domains

(a) $0 < |z-1| < 2$.
(b) $1 < |z| < 3$.

$$\frac{z}{(z-1)(z-3)} = \frac{-1}{(z-1)} + \frac{3}{(z-3)} \quad 3$$

(a)
$$-\frac{1}{z-1} + \frac{3}{z-3} = -\frac{1}{z-1} - \frac{3}{2-(z-1)}$$

$$= -\frac{1}{z-1} - \frac{3}{2}\left(\frac{1}{1-\frac{z-1}{2}}\right) \quad 3$$

if $\left|\frac{z-1}{2}\right| < 1$
$$= -\frac{1}{z-1} - \frac{3}{2}\left(1 + \left(\frac{z-1}{2}\right) + \left(\frac{z-1}{2}\right)^2 + \cdots\right)$$

$$= -\frac{1}{z-1} - \frac{3}{2}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n} \quad 3$$

(b)
$$\frac{-1}{z}\left(\frac{1}{1-1/z}\right) - \frac{1}{\left(1-\frac{z}{3}\right)} \quad 3$$

$|1/z| < 1$ $\quad$ $|z/3| < 1$

$$= -\frac{1}{z}\left(1 + 1/z + 1/z^2 + 1/z^3 + \cdots\right) - \left(1 + z/3 + (z/3)^2 + \cdots\right)$$

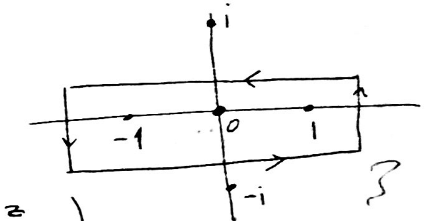
3

Q8. Evaluate the integral,

$$\oint_C \left(\frac{e^z}{z^6 - z^2}\right) dz$$

along the rectangle defined by $x = -2, x = 2, y = -\frac{1}{2}, y = \frac{1}{2}$.

$$\oint \frac{e^z}{z^2(z^4-1)}$$



3

$$\mathrm{Res}(f,0) = \lim_{z\to 0}\frac{d}{dz}\left(\frac{e^z}{z^4-1}\right)$$

$$= \lim_{z\to 0}\frac{e^z(z^4-1) - e^z 4z^3}{(z^4-1)^2} = -1 \quad 3$$

$$\mathrm{Res}(f,1) = \lim_{z\to 1}\frac{e^z}{(z+1)(z^2+1)} = \frac{e}{4} \quad 3$$

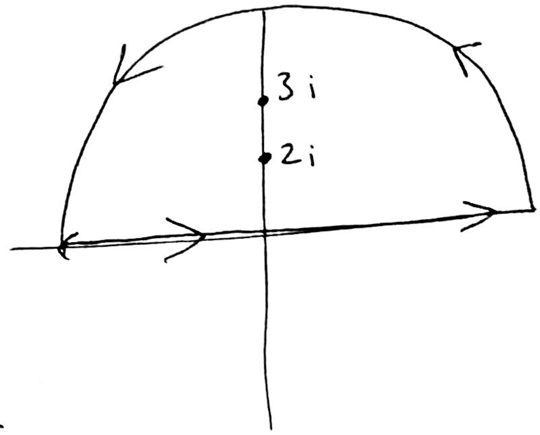
$$\mathrm{Res}(f,-1) = \lim_{z\to -1}\frac{e^z}{(z-1)(z^2+1)} = -\frac{e^{-1}}{4}. \quad 3$$

$$\oint = 2\pi i\left(-1 + \frac{e}{4} - \frac{e^{-1}}{4}\right)$$

3

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+9)(x^2+4)^2}$$

2



$$f(z) = \frac{z^2}{(z^2+9)(z^2+4)^2}$$

$$\mathrm{Res}(f, 3i) = \lim_{z\to 3i} \frac{z^2}{(z+3i)(z^2+4)^2}$$

$$= \frac{-9}{6i\cdot 25} = \frac{9i}{6\cdot 25} = \frac{9i}{150}$$

5

$$\mathrm{Res}(f, 2i) = \lim_{z\to 2i} \frac{d}{dz} \frac{z^2}{(z^2+9)(z+2i)^2}$$

$$= \lim_{z\to 2i} \frac{2z(z^2+9)(z+2i)^2 - z^2\left(2z(z+2i)^2 + (z^2+9\right.}{(z^2+9)^2(z+2i)^4}$$

$$= \frac{4i\cdot 5\cdot(-16) + 4\left(4i(-16) + 5\cdot 2\cdot 4i\right)}{5^2(4i)^4} =$$

5

$$\cancel{= \frac{-160i - 256i + 160i}{5^2\cdot 256} = \ldots}$$

$$\cancel{I = \frac{2\pi i}{2}\left(\frac{9i}{180} - \frac{i}{2\ldots}\right) = }$$

$$\frac{2\pi i}{2}\left(\mathrm{Res}(f,3i) + \mathrm{Res}(f,2i)\right)$$

$$= \frac{\pi}{200}$$ //

3