# King Fahd University of Petroleum & Minerals
## Department of Mathematics and Statistics
## MATH 301
## EXAM II
## 2016 (153)

**Saturday, August 13, 2016** **Allowed Time: 2 Hours**

**Name:** ______________________________________________

**ID Number:** ______________________ **Serial Number:** ______________

**Section Number:** ____________ **Instructor's Name:** ________________

**Instructions**:

1. Write neatly and legibly. You may lose points for messy work.

2. **Show all your work.** No points for answers without justification.

3. **Calculators and Mobiles are not allowed**.

4. Make sure that you have 6 different problems (7 pages + cover page).

| Problem No. | Points | Maximum Points |
|---|---|---|
| 1 | | 15 |
| 2 | | 27 |
| 3 | | 13 |
| 4 | | 15 |
| 5 | | 11 |
| 6 | | 19 |
| Total: | | 100 |

Q1. Evaluate each of the following:

a) $\mathcal{L}\{\sqrt{t}\,e^{-7t}\} = \mathcal{L}\{\sqrt{t}\} \circ (s+7)$ 2

$= \frac{\sqrt{\pi}}{2} s^{-\frac{3}{2}} \circ (s+7)$ 1

$= \frac{\sqrt{\pi}}{2} (s+7)^{-\frac{3}{2}}$ 1

b) $\mathcal{L}\{u(t-1)(t-1)\} = e^{-s}\mathcal{L}\{t\}$ 2

$= e^{-s} \cdot \frac{1}{s^2}$ 1

$= e^{-s}/s^2$

c) $\mathcal{L}\{t^2\cosh 3t\} = (-1)^2 \frac{d^2}{ds^2}\mathcal{L}\{\cosh 3t\}$ 2

$= \frac{d^2}{ds^2} \frac{s}{s^2-9}$ 1

$= \frac{2s^3+54s}{(s^2-9)^3}$ 1

d) $\mathcal{L}\{|\sin t|\} = \frac{1}{1-e^{-\pi s}} \int_0^{\pi} e^{-st}\sin t\, dt$ 2

But $I = \int_0^{\pi} e^{-st}\sin t\, dt$

$= \left[-e^{-st}\cos t - s e^{-st}\sin t\right]_0^{\pi} - s^2 I$

so $I = \frac{1+e^{-\pi s}}{1+s^2}$ 2

Then

$\mathcal{L}\{|\sin t|\} = \frac{1+e^{-\pi s}}{(1-e^{-\pi s})(1+s^2)}$

Q2. Evaluate each of the following:

a) $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+\pi^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+\pi^2}\right\} \circ (t-1) \cdot \mathcal{U}(t-1)$ 2

$= \frac{1}{\pi} \sin \pi t \circ (t-1) \cdot \mathcal{U}(t-1)$ 1

$= \frac{1}{\pi} \sin \pi(t-1) \cdot \mathcal{U}(t-1)$ 1

$= -\frac{1}{\pi} \sin \pi t \cdot \mathcal{U}(t-1)$

b) $\mathcal{L}^{-1}\left\{\frac{6s+3}{s^4+5s^2+4}\right\} = \mathcal{L}^{-1}\left\{2\frac{s}{s^2+1} + \frac{1}{s^2+1} - 2\cdot\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4}\right\}$

1 1 1 1

$= 2\cos t + \sin t - 2\cos 2t - \frac{1}{2}\sin 2t$

1 1 1 1

c) $\mathcal{L}^{-1}\left\{\frac{2s+5}{s^2+6s+34}\right\} = \mathcal{L}^{-1}\left\{\frac{2s+5}{(s+3)^2+25}\right\}$ 1

$= \mathcal{L}^{-1}\left\{\left(\frac{2s-1}{s^2+25}\right)\circ(s+3)\right\}$ 1

$= e^{-3t}\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2+25}\right\}$ 2

$= e^{-3t}\mathcal{L}^{-1}\left\{2\frac{s}{s^2+25} - \frac{1}{5}\frac{5}{s^2+25}\right\}$ 1 1

$= e^{-3t}\left[2\cos 5t - \frac{1}{5}\sin 5t\right]$ 1 1

d) $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\cdot\frac{1}{s^2+1}\right\}$ 1

$= \sin t * \sin t$ 2

$= \int_0^t \sin\tau \sin(t-\tau)\,d\tau$ 1

$= \frac{1}{2}\int_0^t \left(\cos(2\tau-t) - \cos t\right)d\tau$ 1

$= \frac{1}{2}\left[\frac{1}{2}\sin(2\tau-t) - \tau\cos t\right]_{\tau=0}^{t}$ 1

$= \frac{1}{2}\left[\frac{1}{2}\sin t - t\cos t + \frac{1}{2}\sin t\right]$

$= \frac{1}{2}\left[-t\cos t + \sin t\right]$ 1

Q3. Use the Laplace transform to solve the integrodifferential equation

$$y'' + y + \int_0^t y(\tau)\sinh(t-\tau)\,d\tau = \delta(t), \qquad y(0)=1, \qquad y'(0)=0.$$

$$y'' + y + y * \sinh t = \delta(t)$$

Let $\mathcal{L}\{y(t)\} = Y(s)$, then

$$\underbrace{s^2 Y(s) - s\,y(0) - y'(0)}_{2} + Y(s) + \underbrace{Y(s)\cdot\frac{1}{s^2-1}}_{2} = \underbrace{1}_{1}$$

$$Y(s)\left(s^2+1+\frac{1}{s^2-1}\right) = 1+s$$

$$Y(s)\left(\frac{s^4}{s^2-1}\right) = 1+s$$

$$Y(s) = \frac{(1+s)(s^2-1)}{s^4}$$

$$= \frac{s^2-1+s^3-s}{s^4}$$

$$= \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s} - \frac{1}{s^3} \quad 4$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = t - \frac{1}{6}t^3 + 1 - \frac{1}{2}t^2$$

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Q4. Let $\left\{1, \cos\frac{n\pi}{5}x\right\}, n = 1, 2, 3, \ldots,$ be an infinite set of functions.

(a) Show that the set is orthogonal on the interval $[0,5]$.

(i) $\left(1, \cos\frac{n\pi x}{5}\right) = \int_0^5 \cos\frac{n\pi x}{5}\,dx = \frac{5}{n\pi}\sin\frac{n\pi x}{5}\Big]_0^5 = 0$ (2) (1)

for any $n>0$

(ii) $\left(\cos\frac{n\pi x}{5}, \cos\frac{m\pi x}{5}\right) = \int_0^5 \cos\frac{n\pi x}{5}\cos\frac{m\pi x}{5}\,dx$ (2)

$= \frac{1}{2}\int_0^5 \left[\cos\frac{(n+m)\pi x}{5} + \cos\frac{(n-m)\pi x}{5}\right]dx$ (1)

$= \frac{1}{2}\left[\frac{5}{(n+m)\pi}\sin\frac{(n+m)\pi x}{5} + \frac{5}{(n-m)\pi}\sin\frac{(n-m)\pi x}{5}\right]_0^5$ (1)

$= \frac{1}{2}\left[\frac{5}{(n+m)\pi}\sin(n+m)\pi + \frac{5}{(n-m)\pi}\sin(n-m)\pi\right]$ (1)

$= 0$ for $n \neq m$

(b) Find the norm of each function in the set.

$\|1\| = \sqrt{(1,1)} = \sqrt{\int_0^5 dx} = \sqrt{5}$ (2)

$\left\|\cos\frac{n\pi x}{5}\right\|^2 = \int_0^5 \cos^2\frac{n\pi x}{5}\,dx$ (2)

$= \frac{1}{2}\int_0^5 \left[1 + \cos\frac{2n\pi x}{5}\right]dx$ (1)

$= \frac{1}{2}\left[x + \frac{5}{2n\pi}\sin\frac{2n\pi x}{5}\right]_0^5$ (1)

$= \frac{1}{2}\left[5 + \frac{5}{2n\pi}\sin 2n\pi\right]$

$= \frac{5}{2}$ (1)

Then $\left\|\cos\frac{n\pi x}{5}\right\| = \sqrt{\frac{5}{2}}$ for $n>0$.

Q5. Given $f(x)=\begin{cases}0, & -\pi<x<0\\ e^{x}, & 0\le x<\pi\end{cases}$.

(a) Find the Fourier series expansion of the function.

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$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx+b_n\sin nx, \text{ where}$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx=\frac{1}{\pi}\int_0^{\pi} e^{x}\,dx \quad 1$$

$$=\frac{1}{\pi}\left[e^{\pi}-1\right] \quad 1$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{\pi} e^{x}\cos nx\,dx \quad 1$$

$$=\frac{1}{\pi}\,\frac{e^{\pi}(-1)^n-1}{n^2+1} \quad 2$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{\pi} e^{x}\sin nx\,dx \quad 1$$

$$=\frac{n}{\pi}\,\frac{1-e^{\pi}(-1)^n}{n^2+1} \quad 2$$

(b) What does the series converge to when $x=0$?

$x=0\in(-\pi,\pi)$ at which $f$ is discontinuous

Then the Fourier series at $x=0$ converges

to $\dfrac{f(0^+)+f(0^-)}{2}$ 1 $=\dfrac{1+0}{2}=\dfrac{1}{2}$ 1

Q6. Expand $f(x)=x^2,\ 0<x<1$.

(a) in a cosine series,

$$a_0=\frac{2}{L}\int_0^L f(x)\,dx \overset{1}{=} 2\int_0^1 x^2\,dx=\frac{2}{3} \quad 1$$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx=2\int_0^1 x^2\cos n\pi x\,dx \quad 1$$

$$=\frac{4(-1)^n}{n^2\pi^2} \quad 2$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\pi x \quad 1$$

(b) in a sine series,

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx=2\int_0^1 x^2\sin n\pi x\,dx \quad 1$$

$$=\frac{2(-1)^{n+1}}{n\pi}+\frac{4\left[(-1)^n-1\right]}{n^3\pi^3} \quad 2$$

$$f(x)=\sum_{n=1}^{\infty}b_n\sin n\pi x \quad 1$$

(c) in a Fourier series.

$$a_0=\frac{2}{L}\int_0^L f(x)\,dx \overset{1}{=} 2\int_0^1 x^2\,dx=\frac{2}{3} \quad 1$$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}\,dx \quad 1$$

$$=2\int_0^1 x^2\cos 2n\pi x\,dx=\frac{1}{n^2\pi^2} \quad 2$$

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}\,dx \quad 1$$

$$=2\int_0^1 x^2\sin 2n\pi x\,dx=-\frac{1}{n\pi} \quad 2$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos 2n\pi x+b_n\sin 2n\pi x \quad 1$$