# King Fahd University of Petroleum & Minerals
# Department of Mathematics and Statistics
# MATH 301
# EXAM I
# 2016 (153)

**Saturday, July 30, 2016** **Allowed Time: 2 Hours**

**Name:** ______________________________

**ID Number:** ____________________ **Serial Number:** ______________

**Section Number:** ____________ **Instructor's Name:** ________________

**Instructions**:

1. Write neatly and legibly. You may lose points for messy work.

2. **Show all your work.** No points for answers without justification.

3. **Programmable Calculators and Mobiles are not allowed.**

4. Make sure that you have 8 different problems (8 pages + cover page).

| Problem No. | Points | Maximum Points |
|---|---|---|
| 1 | | 10 |
| 2 | | 11 |
| 3 | | 12 |
| 4 | | 16 |
| 5 | | 12 |
| 6 | | 9 |
| 7 | | 19 |
| 8 | | 11 |
| Total: | | 100 |

Q1. Let $G(x,y,z) = x^2z^{-2} - y^2z^{-2}$. Find the directional derivative of $G(x,y,z)$ at the point $(2,4,-1)$ in the direction of $\boldsymbol{v} = \mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$.

$$D_{\mathbf{v}}G = \nabla G \cdot \frac{\mathbf{v}}{|\mathbf{v}|} \quad 2$$

$$\nabla G = \left(2xz^{-2},\ -2yz^{-2},\ -2z^{-3}(x^2-y^2)\right) \quad 3$$

$$\nabla G(2,4,-1) = (4,\ -8,\ -24) \quad 2$$

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{6}}(1,-2,1) \quad 2$$

$$\therefore\quad D_{\mathbf{v}}G = \frac{1}{\sqrt{6}}(4+16-24) \quad 1$$

$$= \frac{-4}{\sqrt{6}} = \frac{-2\sqrt{6}}{3}$$

Q2. Find the length of the curve given by $r(t) = e^t \cos 2t\ \mathbf{i} + e^t \sin 2t\ \mathbf{j} + e^t\ \mathbf{k}$, $\mathbf{0} \leq t \leq \mathbf{3\pi}$.

$$\text{Length} = \int_C ds = \int_0^{3\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \quad 3$$

$$\frac{dx}{dt} = e^t \cos 2t - 2e^t \sin 2t \quad 1$$

$$\frac{dy}{dt} = e^t \sin 2t + 2e^t \cos 2t \quad 1$$

$$\frac{dz}{dt} = e^t \quad 1$$

$$\because \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 6e^{2t} \quad 3$$

$$\therefore \text{length} = \int_0^{3\pi} \sqrt{6}\, e^t\, dt$$

$$= \sqrt{6}\, e^t \Big]_0^{3\pi} \quad 1$$

$$= \sqrt{6}\left[e^{3\pi} - 1\right] \quad 1$$

Q3. On the unit sphere, the gravitational attraction force between a mass $m_1$ at the origin and a mass $m_2$ at the sphere is given by the vector field $\mathbf{F} = -\boldsymbol{c}(y\,z\,\mathbf{i} + x\,z\,\mathbf{j} + x\,y\,\mathbf{k})$ where $\boldsymbol{c}$ is a positive constant and $(x, y, z)$ is the coordinate of $m_2$.

i) Show that F is conservative.

$$\text{curl } F = -c\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = -c\,(x - x,\ y - y,\ z - z) \quad 2 \quad 1$$

$$= (0, 0, 0) \quad 1$$

$\therefore$ F is conservative

ii) Verify that $\emptyset(x, y, z) = -\boldsymbol{c}\,xyz$ is a potential of the vector field $\mathbf{F}$.

$$\phi_x = -c\,yz = P \quad 1$$

$$\phi_y = -c\,xz = Q \quad 1$$

$$\phi_z = -c\,xy = R \quad 1 \qquad \therefore F = \nabla\phi$$

iii) Evaluate the work done by the force $\boldsymbol{F}$ in moving a mass $m_2$ from the point $A(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$ to the point $B(1,0,0)$.

$$\text{Work} = \int_A^B F \cdot dr \quad 2$$

$$= \phi(1,0,0) - \phi\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{\sqrt{2}}\right) \quad 2$$

$$= 0 + c\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{\sqrt{2}}\right)$$

$$= \frac{c}{4\sqrt{2}} \quad 1$$

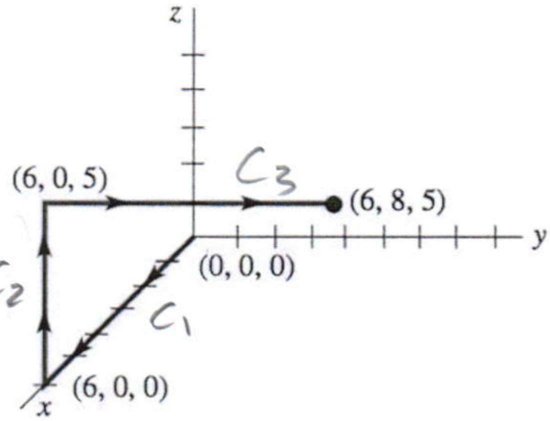
Q4. Evaluate the line integral

$$I = \int_C y\,dx + z\,dy + x\,dz$$

on the given curve between (0,0,0) to (6,8,5).

$C = C_1 \cup C_2 \cup C_3$

where




$C_1$: $x = 6t$, $y = 0$, $z = 0$, $0 \le t \le 1$ (3)

$C_2$: $x = 6$, $y = 0$, $z = 5t$, $0 \le t \le 1$ (3)

$C_3$: $x = 6$, $y = 8t$, $z = 5$, $0 \le t \le 1$ (3)

$$\int_{C_1} y\,dx + z\,dy + x\,dz = 0 \quad (2)$$

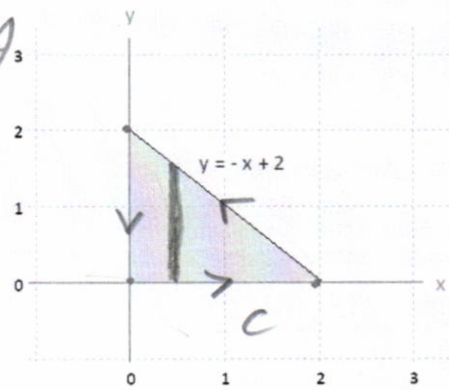
$$\int_{C_2} y\,dx + z\,dy + x\,dz = \int_0^1 6(5)\,dt = 30 \quad (2)$$

$$\int_{C_3} y\,dx + z\,dy + x\,dz = \int_0^1 5(8)\,dt = 40 \quad (2)$$

Then $I = 0 + 30 + 40 = 70$ (1)

Q5. Use **Green's Theorem** to evaluate $\oint_C \mathbf{2y^2dx + x^2y\,dy}$, where $\boldsymbol{C}$ consists of the boundary of the shaded region $\boldsymbol{R}$ with a counterclockwise orientation.

$$I = \oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA$$




2

$$= \iint_R (2xy - 4y)\,dA$$

2 2

$$= \int_0^2 \int_0^{2-x} (2x-4)y \; dy\,dx$$

1 1

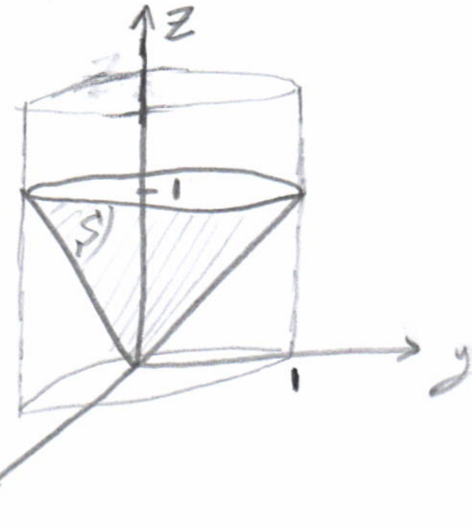
$$= \int_0^2 \left[(2x-4)\frac{y^2}{2}\right]_{y=0}^{2-x} dx \quad 1$$

$$= \int_0^2 (x-2)^3 \, dx \quad 1$$

$$= \left.\frac{(x-2)^4}{4}\right]_{x=0}^{2} \quad 1$$

$$= 0 - 4 = -4 \quad 1$$

Q6. Find the surface area of the portion of the cone $z = \sqrt{x^2 + y^2}$ inside the cylinder $x^2 + y^2 = 1$.



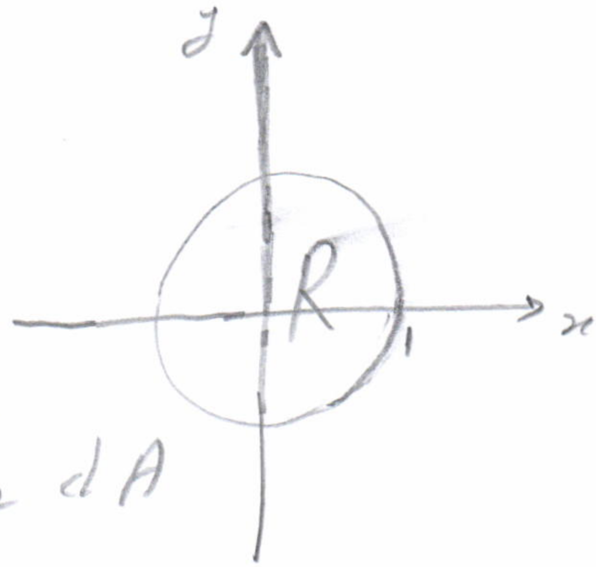

$$\text{Area} = \iint_{S} dS \quad 2$$

$$dS = \sqrt{1 + f_x^2 + f_y^2}\, dA \quad 1$$

where $z = f(x,y) = \sqrt{x^2+y^2}$

$$f_x = \frac{2x}{2\sqrt{x^2+y^2}} \quad 1$$

$$f_y = \frac{2y}{2\sqrt{x^2+y^2}} \quad 1$$




$$\therefore dS = \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}\, dA$$

$$= \sqrt{2}\, dA \quad 2$$

Then $\text{area} = \iint_R \sqrt{2}\, dA \quad 1$

$$= \sqrt{2} \iint_R dA$$

$$= \sqrt{2}\, \pi (1)^2$$

$$= \sqrt{2}\, \pi \quad 1$$

Q7. Use **Stokes' theorem** to evaluate $\oint_C \boldsymbol{z^2x\,dx + z\,e^x\,dy + x\,dz}$ where $\boldsymbol{C}$ is the curve of intersection of the plane $y = 1$ and the sphere $x^2 + y^2 + z^2 = 4$, by finding a surface $S$ with $\boldsymbol{C}$ as its boundary and such that the orientation of $\boldsymbol{C}$ is counterclockwise as viewed from above.

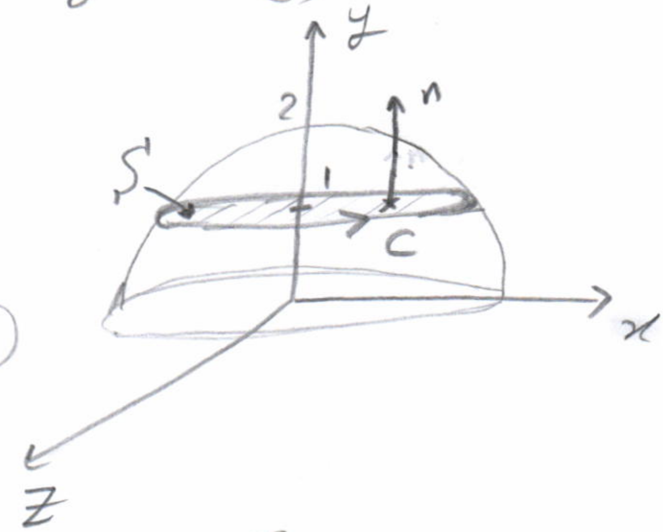
$$I = \oint_C F \cdot dr = \iint_{S'} \text{curl } F \cdot n \, dS' \quad (2)$$

where $F = (z^2x,\ z e^x,\ x)$

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2x & ze^x & x \end{vmatrix} = (e^x,\ 2zx - 1,\ ze^x) \quad (3)$$

$n = \dfrac{\nabla g}{|\nabla g|}$ where $g = y - 1$ (1)

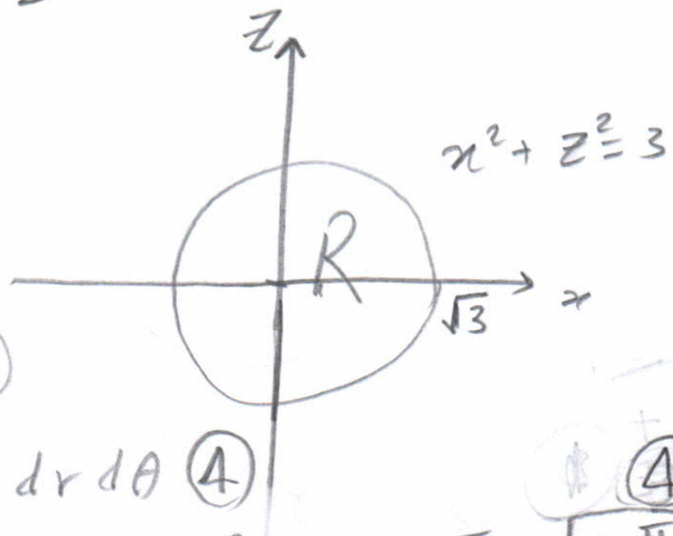
$\therefore\ n = (0, 1, 0)$ (1)




$\text{curl } F \cdot n = 2zx - 1$ (1)

$\left.\begin{array}{l} dS' = \sqrt{1 + f_x^2 + f_z^2}\, dA \\ \text{where } y = f(x, z) = 1 \end{array}\right\}$ (1)

So, $dS' = dA$ (1)




Then

$I = \iint_R (2zx - 1)\, dA$ (1)

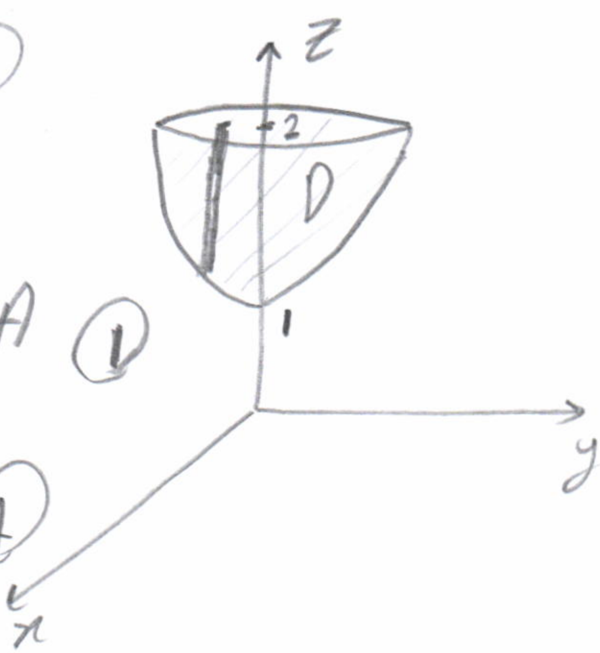
$= \int_0^{2\pi}\int_0^{\sqrt{3}} (2r^2 \sin\theta\cos\theta - 1)\, r\, dr\, d\theta$ (4) (4)

$= 2\int_0^{\sqrt{3}} r^3\, dr \int_0^{2\pi} \sin\theta\cos\theta\, d\theta - \pi(\sqrt{3})^2 = 0 - 3\pi = \boxed{-3\pi}$

Q8. Let D be the region bounded by the paraboloid $z = x^2 + y^2 + 1$ and the plane $z = 2$. Use the **divergence theorem** to find the outward flux $\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS$ of the vector field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + \mathbf{k}$, where S is the boundary of D.

$$I = \iint_S F\cdot n\, dS = \iiint_D \operatorname{div} F\, dV \quad (2)$$

$$I = \iiint_D 2\, dV \quad (1)$$

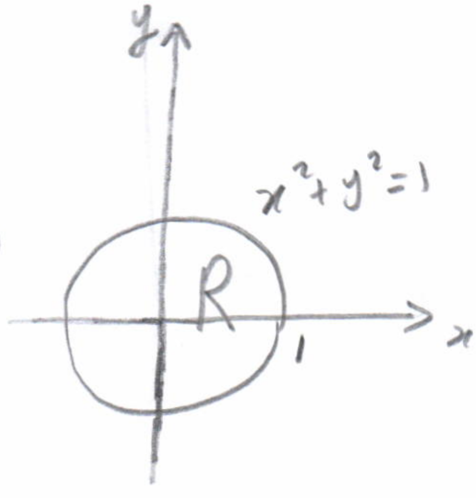



$$= 2\iint_R \left[2 - (x^2+y^2+1)\right] dA \quad (1)$$

$$= 2\int_0^{2\pi}\int_0^1 (1 - r^2)\, r\, dr\, d\theta \quad (4)$$

$$= 2\int_0^{2\pi} d\theta \int_0^1 (r - r^3)\, dr$$

$$= 2(2\pi)\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 \quad (2)$$




$$= 4\pi\left[\frac{1}{2} - \frac{1}{4}\right]$$

$$= \pi \quad (1)$$