1. Apply the eigenvalue method to solve the system $x_1' = 5x_1 - 6x_3$, $x_2' = 2x_1 - x_2 - 2x_3$, $x_3' = 4x_1 - 2x_2 - 4x_3$.

Solution. Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $A = \begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}$, then the system in matrix form is $X' = AX$.

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 0 & -6 \\ 2 & -1-\lambda & -2 \\ 4 & -2 & -4-\lambda \end{vmatrix} = (5-\lambda)\begin{vmatrix} -1-\lambda & -2 \\ -2 & -4-\lambda \end{vmatrix} - 6\begin{vmatrix} 2 & -1-\lambda \\ 4 & -2 \end{vmatrix} = -\lambda\left(\lambda^2 - 1\right).$$

- Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -1$.
- Eigenvectors:

$\lambda_1 = 0$: $\begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 1 & -4 \\ 2 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -2 \\ 2 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, we get 1 free variable. If $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is an eigenvector, then $a + 2b - 2c = 0$, $-5b + 2c = 0$. Take $c = 5$ to get $b = 2$, $a = 6$. So $v_1 = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}$ is an eigenvector corresponding to λ_1.

$\lambda_2 = 1$: $\begin{bmatrix} 4 & 0 & -6 \\ 2 & -2 & -2 \\ 4 & -2 & -5 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & 0 & -6 \\ 1 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 4 & -2 \\ 1 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, 1 free variable. If $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is an eigenvector, then $a - b - c = 0$, $2b - c = 0$, so $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3c/2 \\ c/2 \\ c \end{bmatrix}$, and $v_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ is an eigenvector for λ_2.

$\lambda_3 = -1$: $\begin{bmatrix} 6 & 0 & -6 \\ 2 & 0 & -2 \\ 4 & -2 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 4 & -2 & -3 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, 1 free variable. If $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is an eigenvector, then $a - c = 0$, $-2b + c = 0$, so $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} c \\ c/2 \\ c \end{bmatrix}$, and $v_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ is an eigenvector for λ_3.

The general solution of the system is $X = c_1 \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix} + c_2 e^t \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, i.e.:
$x_1 = 6c_1 + 3c_2e^t + 2c_3e^{-t}$, $x_2 = 2c_1 + c_2e^t + c_3e^{-t}$, $x_3 = 5c_1 + 2c_2e^t + 2c_3e^{-t}$.

2. The vectors $X_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^t$, $X_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{3t}$, $X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} e^{5t}$ are solutions of the system $X' = AX$, where $A = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}$.

(a) Verify that the vectors X_1, X_2, X_3 are linearly independent and that X_2 is a solution of the system.

(b) Solve the IVP: $X' = AX$, $X(0) = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$.

Solution. (a) Wronskian is $\begin{vmatrix} 2 & -2 & 2 \\ 2 & 0 & -2 \\ 1 & 1 & 1 \end{vmatrix} e^t e^{3t} e^{5t} = 16e^{9t} \neq 0$, so the vectors are linearly independent.

Also, $X_2' = \begin{bmatrix} -6 \\ 0 \\ 3 \end{bmatrix} e^{3t}$ and $AX_2 = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{3t} = \begin{bmatrix} -6e^{3t} \\ 0 \\ 3e^{3t} \end{bmatrix} = X_2'$, hence X_2 is a solution of the system.

(b) We are given that X_1, X_2, X_3 are solutions of the system, so the general solution is $X = c_1X_1 + c_2X_2 + c_3X_3$. Substitution gives: $c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$. This system has solution $c_1 = 1$, $c_2 = 2$, $c_3 = 1$. The IVP solution is: $X = X_1 + 2X_2 + X_3$.