Part II. Solve each of the following 6 questions showing full details.

**Q1. (14 Points)** Find the absolute maximum and the absolute minimum of

$$f(x,y) = 2x + 4y - x^2 - 2y^2$$

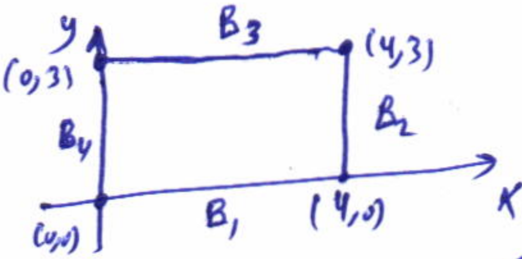
on the rectangular region $R = \{(x,y) : 0 \le x \le 4,\ 0 \le y \le 3\}$.

2 pts

$f_x = 2 - 2x$ | $f_x = 0 \Rightarrow x = 1$

$f_y = 4 - 4y$ | $f_y = 0 \Rightarrow y = 1$

$(1,1)$ is the only critical point.




2 pts

$B_1$ : $y = 0$ , $x \in [0,4]$

$f_1(x) = f(x,0) = 2x - x^2$

$f_1'(x) = 2 - 2x$

$f_1'(x) = 0 \Rightarrow x = 1$

$\boxed{(1,0), (0,0), (4,0)}$

2 pts

$B_2$ : $x = 4$ ; $y \in [0,3]$

$f_2(y) = f(4,y) = 4y - 2y^2 - 8$

$f_2'(y) = 4 - 4y$.

$f_2'(y) = 0 \Rightarrow y = 1$

$\boxed{(4,0), (4,1), (4,3)}$

2 pts

$B_3$ : $y = 3$, $x \in [0,4]$

$f_3(x) = f(x,3) = 2x - x^2 - 6$

$f_3'(x) = 2 - 2x$

$f_3'(x) = 0 \Rightarrow x = 1$

$\boxed{(0,3), (1,3), (4,3)}$

2 pts

$B_4$ : $x = 0$ , $y \in [0,3]$

$f_4(y) = 4y - 2y^2$

$f_4'(y) = 4 - 4y$

$f_4'(y) = 0 \Rightarrow y = 1$

$\boxed{(0,0), (0,1), (0,3)}$

Candidates

2 pts

| $(x,y)$ | $f(x,y)$ |
|---|---|
| $(1,1)$ | 3 |
| $(1,0)$ | 1 |
| $(0,0)$ | 0 |
| $(4,0)$ | $-8$ |
| $(4,1)$ | $-6$ |
| $(4,3)$ | $-14$ |
| $(0,3)$ | $-6$ |
| $(1,3)$ | $-5$ |
| $(0,1)$ | 2 |

$\therefore$ absolute maximum is 3 at $(1,1)$ & absolute minimum is $-14$ at $(4,3)$

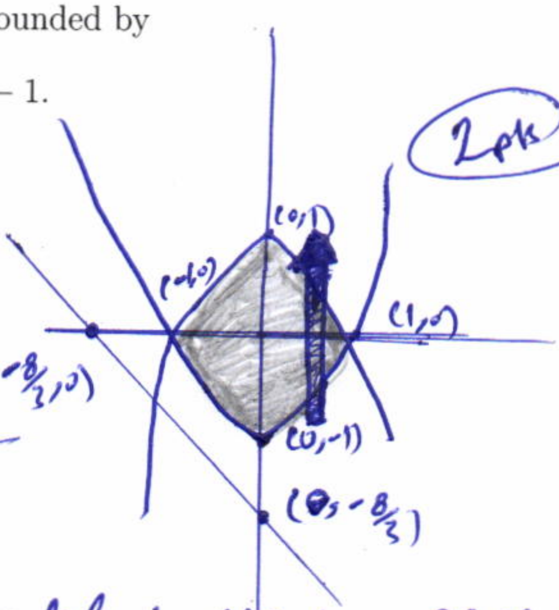
2 pts

**Q3. (10 Points)** Find the volume of the solid between the two planes

$$x+y+z=2,\ z-2x-2y=10$$

that lies above region in the $xy$-plane bounded by

$$y=1-x^2,\ y=x^2-1.$$




2pts

- The two parabolas in the $xy$-plane intersect when: $y = 1-x^2 = x^2-1$

1pt

$2x^2 = 1$

$x^2 = 1,\ x = \mp 1,$

i.e. at $(-1,0)$ & $(1,0)$.

- We check whether $z_1 = 2-(x+y)$ & $z_2 = 10+2(x+y)$ intersect over the region $D$ bounded by the two parabolas.

$z_1 = z_2 \Leftrightarrow 2-(x+y) = 10+2(x+y) \Leftrightarrow x+y = -\frac{8}{3}$.

1pt

Clearly, the line $x+y=-\frac{8}{3}$ does not pass through $D$. So, $z_1$ & $z_2$ do not intersect over $D$.

- $V$ = Volume $= \iint_D |z_2 - z_1|\, dA = \left|\iint_D (z_2-z_1)\, dA\right|$

1pt 1pt

since $z_1, z_2$ do not intersect over $D$

2pts

$V = \left|\int_{-1}^{1}\int_{x^2-1}^{1-x^2} \left(10+2x+2y-(2-x-y)\right) dy\, dx\right|$

$= \left|\int_{-1}^{1}\int_{x^2-1}^{1-x^2} \left((8+3x)+3y\right) dy\, dx\right| \overset{1pt}{=} \left|\int_{-1}^{1}\left((8+3x)y + \frac{3y^2}{2}\Big|_{x^2-1}^{1-x^2}\right) dx\right|$

zero

$= \left|\int_{-1}^{1} \left((8+3x)(1-x^2-x^2+1)\right) dx\right| = \left|\int_{-1}^{1} (16-16x^2+6x-6x^3)\, dx\right|$

1pt $= \left|\left(16x - \frac{16x^3}{3} + \left(\frac{6x^2}{2} - \frac{6x^4}{4}\right)\right)\Big|_{-1}^{1}\right|$

zero

$= 16(1-(-1)) - \frac{16}{3}(1-{-1}) = 32 - \frac{32}{3} = \boxed{\frac{64}{3}}$

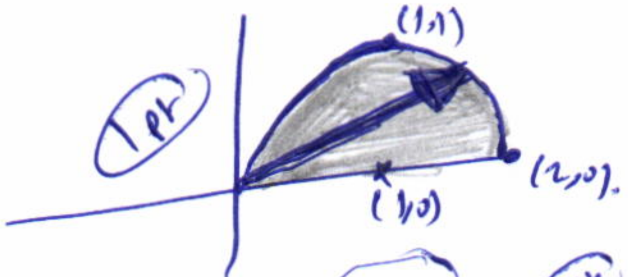
**Q4. (10 Points)** Convert the integral

$$I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx$$

to polar coordinates, then evaluate it.

$0 \leq y \leq \sqrt{2x - x^2}$ ; $0 \leq y^2 \leq 2x - x^2$.

$(x-1)^2 + y^2 = 1$ ; $y \geq 0$ $\quad x^2 + y^2 \leq 2x$ , $y \geq 0$.




$x^2 + y^2 = 2x$

$r^2 = 2r\cos\theta$

$\boxed{r = 2\cos\theta}$ 1pt

1pt 1pt $0 \leq \theta \leq \frac{\pi}{2}$

$I = \int_0^{\pi/2} \int_0^{2\cos\theta} r \cdot \boxed{r}\, dr\, d\theta$

Jacobian

2pts

1pt $= \int_0^{\pi/2} \left( \frac{r^3}{3} \Big|_0^{2\cos\theta} \right) d\theta = \frac{1}{3} \int_0^{\pi/2} 8\cos^3\theta \, d\theta$

1pt $= \frac{8}{3} \int_0^{\pi/2} \cos\theta \,(1 - \sin^2\theta)\, d\theta$

1pt $= \frac{8}{3} \int_0^{\pi/2} (\cos\theta - \sin^2\theta \cdot \cos\theta)\, d\theta$

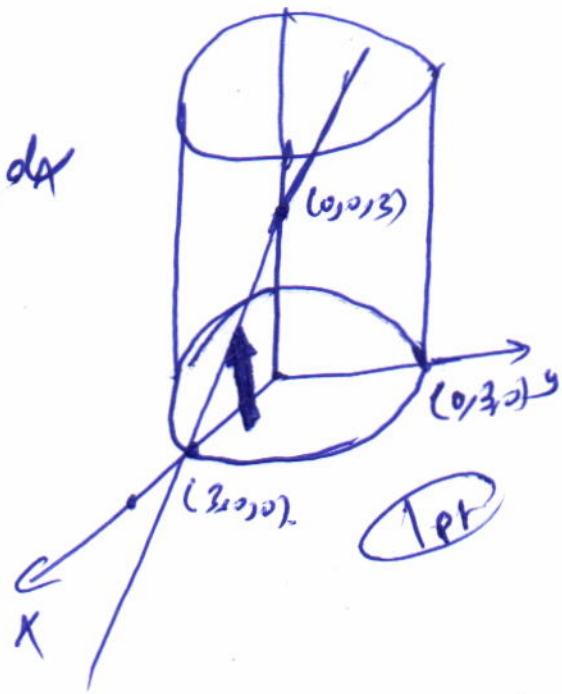
1pt $= \frac{8}{3} \left( \left( \sin\theta - \frac{\sin^3\theta}{3} \right) \Big|_0^{\pi/2} \right)$

$= \frac{8}{3} \left( \left(1 - \frac{1}{3}\right) - 0 \right) = \boxed{\frac{16}{9}}$

**Q5. (12 Points)** Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $x + z = 3$.

4 pt

$$V = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{0}^{3-x} dz\, dy\, dx$$




1pt

1pt

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} z\Big|_{0}^{3-x} dy\, dx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (3-x)\, dy\, dx$$

the shadow in the xy-plane is the circle

$x^2 + y^2 = 9$

$\boxed{r = 3}$


Jacobian

2 pt

$$= \int_{0}^{2\pi} \int_{0}^{3} (3 - r\cos\theta)\, \boxed{r}\, dr\, d\theta$$

1pt

$$= \int_{0}^{2\pi} \left( \left( \frac{3r^2}{2} - \frac{r^3}{3}\cos\theta \right)\Big|_{0}^{3} \right) d\theta$$

1pt

$$= \int_{0}^{2\pi} \left( \frac{27}{2} - 9\cos\theta \right) d\theta$$

1pt

$$= \left( \frac{27}{2}\theta - 9\sin\theta \right)\Big|_{0}^{2\pi}$$

$$= 27\pi - 0$$

$$= \boxed{27\pi}$$

1pt

half

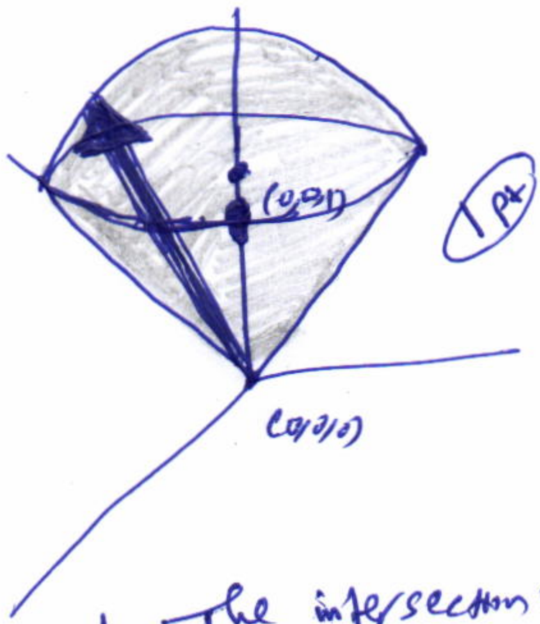
**Q6.** **(10 Points)** Find the volume of the solid below the sphere $x^2+y^2+z^2=2z$ and above the cone $z=\sqrt{x^2+y^2}$.

$x^2+y^2+(z-1)^2=1$

sphere with center $(0,0,1)$ & radius 1.

$z=\sqrt{x^2+y^2}$

half cone with vertex $(0,0,0)$ & opening towards the true z-axis.




1 pt

Sphere:

$x^2+y^2+z^2=2z$

$\rho^2=2\rho\cos\phi$

$\boxed{\rho=2\cos\phi}$ 1 pt

half cone:

$z=\sqrt{x^2+y^2}=r$

$\frac{z}{r}=1$

$\tan\phi=1$ ; $\phi=\frac{\pi}{4}$ 1 pt

The intersection:

$z^2=x^2+y^2$

$=2z-(x^2+y^2)$

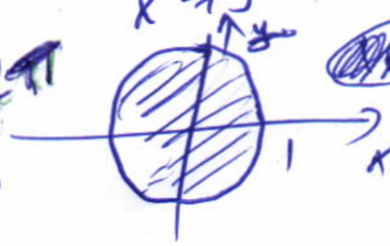
$2(x^2+y^2)=2z$

$x^2+y^2=z$

$z^2=x^2+y^2=z$

$z^2=z$ ; $z=0,1$

the shadow in the xy-plane is the circle $x^2+y^2=1$




Volume $= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$ 3 pts 1 pt

15

1 pt $=\int_0^{2\pi}\int_0^{\pi/4}\left(\frac{\rho^3}{3}\sin\phi\Big|_0^{2\cos\phi}\right)d\phi\, d\theta$

$=\int_0^{2\pi}\int_0^{\pi/4}\frac{8}{3}\cos^3\phi\sin\phi\, d\phi\, d\theta$

1 pt $=\frac{8}{3}\int_0^{2\pi}\left(-\frac{\cos^4\phi}{4}\Big|_0^{\pi/4}\right)d\theta=-\frac{8}{12}\left(\frac{1}{4}-1\right)2\pi$

$=\boxed{\frac{2\pi}{2}}=\pi$ 1 pt