King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 201: Second Major Exam, Summer 153 (120 minutes)

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**Name:** **Student ID:**

**Section:** **Serial Number:**

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Solve all problems. Show **full details** of your solution.

| Question | Grade |
|---|---|
| 1 | /18 |
| 2 | /11 |
| 3 | /10 |
| 4 | /5 |
| 5 | /8 |
| 6 | /10 |
| 7 | /14 |
| 8 | /10 |
| 9 | /6 |
| 10 | /8 |
| TOTAL | |

**Q1. (a) (5 points)** Find the distance between the plane $2x - 4y + 4z = 12$ and the plane $x - 2y + 2z = 3$.

(1pt) • Notice that the two planes are parallel since $\vec{n_1} = \langle 2, -4, 4 \rangle$ ; $\vec{n_2} = \langle 1, -2, 2 \rangle$ ; $\vec{n_1} = 2\vec{n_2}$.

(1pt) • Take the point $(6, 0, 0)$ on the first plane.

• Set $x - 2y + 2z - 3 = 0$ $(Ax + By + Cz + d = 0)$.

Distance $\overset{(1pt)}{=} \dfrac{|Ax_0 + By_0 + Cz_0 + d|}{\sqrt{A^2 + B^2 + C^2}} \overset{(1pt)}{=} \dfrac{|6 + 0 + 0 - 3|}{\sqrt{1 + 4 + 4}} = \dfrac{3}{3} \overset{(1pt)}{=} 1$

**(b) (6 points)** Find the symmetric equations for the line through $(2, 1, -1)$ and perpendicular to both of $\vec{v} = \langle 1, 1, 0 \rangle$ and $\vec{w} = \langle 0, 1, 1 \rangle$.

• A vector parallel to the line is

$$\vec{u} \overset{(2pt)}{=} \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} \overset{(2pt)}{=} \langle 1, -1, 1 \rangle$$

• The symmetric equations for the line are:

$$\frac{x-2}{1} \overset{(1pt)}{=} \frac{y-1}{-1} \overset{(1pt)}{=} \frac{z+1}{1} \;; \quad \boxed{x - 2 = 1 - y = z + 1}$$

**(c) (7 points)** Find the equation of the plane that passes through the points $(0, 1, -1)$ and $(-1, 2, 1)$ and is perpendicular to the plane $x + 2y - 3z = 5$.

• Consider $P(0, 1, -1)$ & $Q(-1, 2, 1)$.

(2pt) • A vector in the plane is

$$\vec{PQ} = \langle -1, 1, 2 \rangle$$

(1pt) • A normal vector to the given plane is $\vec{n} = \langle 1, 2, -3 \rangle$.

• A normal vector to the required plane is

$$\vec{N} \overset{(1pt)}{=} \vec{PQ} \times \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 2 \\ 1 & 2 & -3 \end{vmatrix} \overset{(1pt)}{=} \langle -7, -1, -3 \rangle$$

(2pt) • Using $P$ & $\vec{N}$, the eqn. of the required plane is

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$-7(x - 0) - 1(y - 1) - 3(z + 1) = 0$$

$$\boxed{-7x - y - 3z = 2} \quad \boxed{7x + y + 3z = -2}$$

**Q2.** Consider

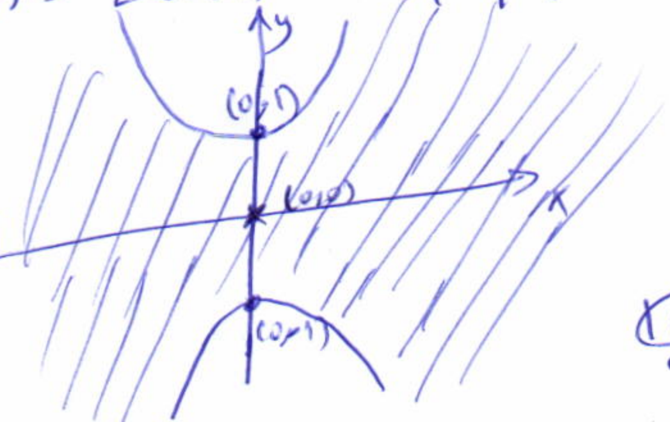
$$f(x,y) = \sqrt{1 + x^2 - y^2}$$

**(a) (5 points)** Find and Sketch the *domain* of $f$.

1pt

$D(f) = \{ (x,y) \in \mathbb{R}^2 \mid 1 + x^2 - y^2 \geq 0 \}$

3 pts




$1 + x^2 - y^2 = 0.$

$y^2 - x^2 = 1$

hyperbola with vertices

$(0, \mp 1)$

1pt

- check point: $(0,0)$: $1 - 0 - 0 \geq 0$ ✓

**(b) (6 points)** Identify (name) and sketch the *graph* of $f$.

2pt

- $z = \sqrt{1 + x^2 - y^2}$

$z^2 = 1 + x^2 - y^2$ ; $z \geq 0$

$y^2 + z^2 - x^2 = 1$

$z \geq 0$

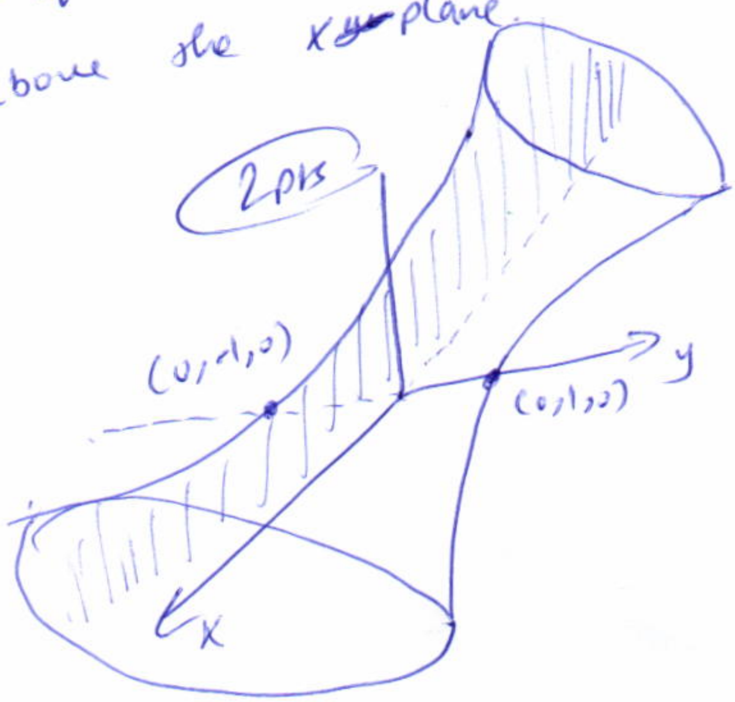
1pt

- $y^2 + z^2 - x^2 = 1$ represents a hyperboloid of one sheet with axis the $x$-axis.

1pt

- Since $z \geq 0$, the function represents the upper ~~half~~ of the hyperboloid which lies above the ~~xy~~-plane

2pts



**Q3. (10 points)** Find the limit (if it exists) or show that it does not exist:

**(a)** $\lim_{(x,y)\to(0,0)} \frac{xy^4}{2x^2+y^8}$ $= L$

2pts $\boxed{\text{Let } x = my^4}$

$L = \lim_{y\to 0} \frac{my^4 \cdot y^4}{2(my^4)^2 + y^8} = \lim_{y\to 0} \frac{\cancel{y^8}(m)}{\cancel{y^8}(2m^2+1)}$ 1pt

$= \frac{m}{2m^2+1}$ (para dependent)

2pts $\therefore$ $L$ Does not exist.

**(b)** $\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$ $= \lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}}$ 3pts

2pts

$= 0$

**Q4 (5 points)** Let $w = \frac{xy}{y^2+2\sin^2 y}$. Find $\frac{\partial^3 w}{\partial x^2 \partial y}$.

1pt Since the function satisfies the hypothesis of Clairaut's Theorem on its domain, we have

$$\frac{\partial^3 w}{\partial x^2 \partial y} = \frac{\partial^3 w}{\partial y \partial x^2} = \frac{\partial}{\partial y \partial x}\left(\frac{\partial w}{\partial x}\right)$$ 2pts

$$= \frac{\partial}{\partial y \partial x}\left(\frac{y}{y^2+2\sin^2 y}\right) = \frac{\partial}{\partial y}(0)$$ 2pts

$$= 0.$$

**Q5. (8 points)** Find the linearization $L(x, y)$ of

$$f(x, y) = e^{2x-3y}$$

at the point $(3, 2)$. Use $L(x, y)$ to estimate $f(2.99, 2.05)$.

2 pts • $f_x = 2e^{2x-3y}$

2 pts • $f_y = -3e^{2x-3y}$

Consider $P(3,2)$

$f(3,2) = 1$

$f_x|_P = 2$ ; $f_y|_P = -3$

2 pts • $L(x,y) = f(a,b) + f_x|_P(x-a) + f_y|_P(y-b)$

$= 1 + 2(x-3) - 3(y-2) = \boxed{2x - 3y + 1}$

2 pts • $f(2.99, 2.05) \approx L(2.99, 2.05) = 2(2.99) - 3(2.05) + 1$

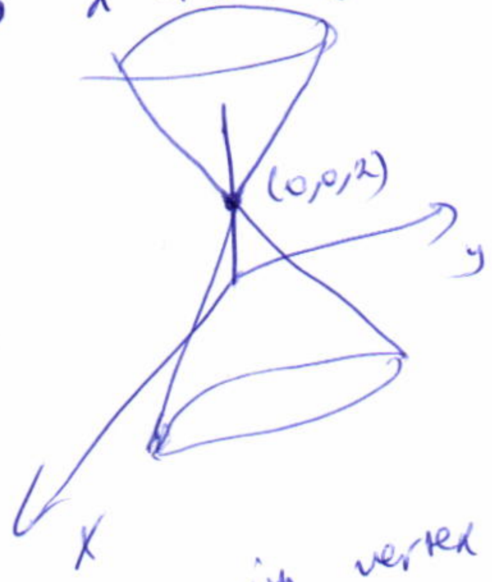
$= 0.83$

**Q6. (10 points)** Identify (name) the following two surfaces

$$x^2 + y^2 = (z-2)^2 \text{ and } x^2 + y^2 = 2 - z$$

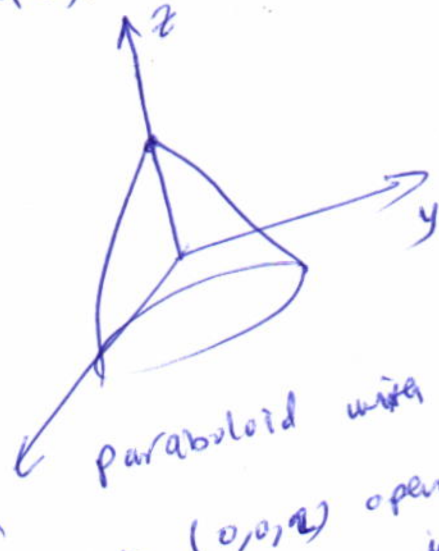
and find their intersection.

• $x^2 + y^2 = (z-2)^2$




2 pts • Cone with vertex $(0,0,2)$ and axis the $z$-axis.

• $2 - (x^2 + y^2) = z$




paraboloid with vertex the point $(0,0,2)$ opening downwards with axis the $z$-axis. 2 pts

• Intersection: $x^2 + y^2 = \boxed{(z-2)^2 = 2 - z}$

$(z-2)(z-2+1) = 0$

$z = 2$ or $\boxed{z = 1}$ 2 pts

• $z = 2$. $x^2 + y^2 = 0$. one point $(0,0,2)$ 2 pts

• $z = 1$ $x^2 + y^2 = 1$ circle in the plane $z = 1$ with center $(0,0,1)$ & radius $1$. 2 pts

**Q7.** Consider the level surface

$$F(x,y,z) = \cos \pi x - x^2 y + e^{xz} + yz = 4.$$

**(a) (10 points)** Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $P(0,1,2)$.

• Using implicit differentiation:

$$-\pi \sin \pi x - 2xy + e^{xz}\left(z + x\frac{\partial z}{\partial x}\right) + y\frac{\partial z}{\partial x} = 0.$$

3 pts

$$\frac{\partial z}{\partial x} = \frac{\pi \sin \pi x + 2xy - z e^{xz}}{x e^{xz} + y} = -\frac{F_x}{F_z}$$

2 pts $\left.\frac{\partial z}{\partial x}\right|_P = \frac{-2}{1} = -2.$

• $\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = -\frac{(x^2+z)}{xe^{xz}+y} = \frac{x^2-z}{xe^{xz}+y}$

3 pts

2 pts $\left.\frac{\partial z}{\partial y}\right|_P = \frac{-2}{1} = -2.$

**(b) (4 points)** Find the equation of the tangent plane to the level surface at $P(0,1,2)$.

$$\vec{\nabla} F = \langle F_x, F_y, F_z \rangle$$
$$= \langle -\pi \sin \pi x - 2xy + ze^{xz},\ z - x^2,\ xe^{xz} + y \rangle$$

2 pts

$$\vec{\nabla} F\big|_P = \langle +2, 2, 1 \rangle$$

Eqn of the tangent plane:

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0.$$

1 pt

$$2(x-0) + 2(y-1) + 1(z-2) = 0$$

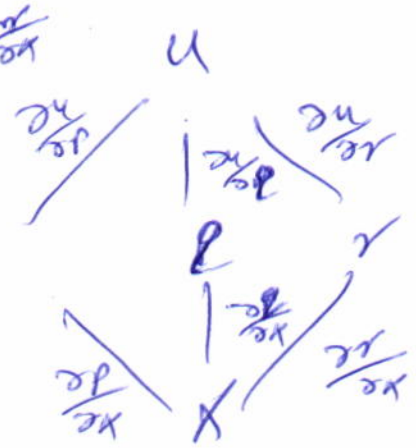
1 pt $\boxed{2x + 2y + z = 4}$

**Q8 (10 points)** Let $u = e^{qr}\sin^{-1} p$, where

$$p = \sin x,\ q = z^2 y \text{ and } r = \frac{1}{z}.$$

Find the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial z}$ at $(x, y, z) = (\frac{\pi}{4}, \frac{1}{2}, -\frac{1}{2})$.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$



3 pts

$$= \frac{e^{qr}}{\sqrt{1-p^2}}\cdot \cos x + r e^{qr}\sin^{-1}p \cdot 0 + q e^{qr}\sin^{-1}p \cdot 0$$

$$\left.\frac{\partial u}{\partial x}\right|_P = \frac{e^{-\frac{1}{4}}}{\cancel{\sqrt{\frac{1}{2}}}}\cdot\cancel{\frac{1}{\sqrt{2}}} + 0 + 0$$

2 pts

$$= e^{-\frac{1}{4}}.$$

$x = \frac{\pi}{4}$, $y = \frac{1}{2}$, $z = -\frac{1}{2}$ | $p = \frac{1}{\sqrt{2}}$, $q = \frac{1}{8}$, $r = -2$

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$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$= \frac{e^{qr}}{\sqrt{1-p^2}}\cdot 0 + r e^{qr}\cdot\sin^{-1}p \cdot 2yz + q e^{qr}\sin^{-1}p \cdot \frac{-1}{z^2}$$

3 pts

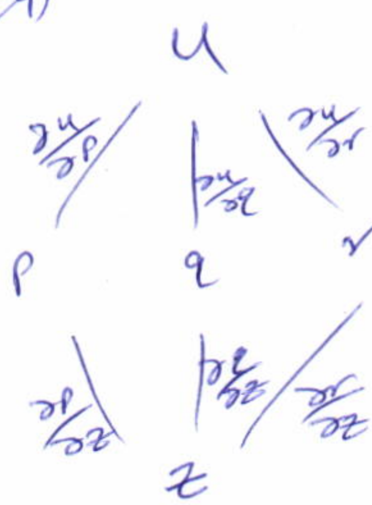
$$\left.\frac{\partial u}{\partial z}\right|_P = 0 + (\cancel{-}2)\, e^{-\frac{1}{4}}\left(\frac{\pi}{4}\right)\cdot\left(\cancel{-}\frac{1}{2}\right).$$

2 pts

$$+ \frac{1}{8}\cdot e^{-\frac{1}{4}}\cdot\frac{\pi}{4}\cdot(-4)$$

$$= e^{-\frac{1}{4}}\left(\frac{\pi}{4} - \frac{\pi}{8}\right)$$

$$= \frac{\pi}{8} e^{-\frac{1}{4}}.$$



**Q9 (6 points)** The directional derivative of $f(x,y)$ at $P$ in the direction of $\overrightarrow{v} =< 1,1 >$ is $2\sqrt{2}$ and in the direction of $\overrightarrow{w} =< 0,-2 >$ is $-3$. Find the directional derivative of $f(x,y)$ at $P$ in the direction of $\overrightarrow{u} =< -1,2 >$.

2pts Suppose that $\nabla f|_P = \langle a,b \rangle$.

1pt $\nabla f|_{P,\vec{v}} = \nabla f|_P \cdot \frac{\vec{v}}{|\vec{v}|}$, so $2\sqrt{2} = \frac{a+b}{\sqrt{2}}$ $\boxed{a+b=4}$

1pt $\nabla f|_{P,\vec{w}} = \nabla f|_P \cdot \frac{\vec{w}}{|\vec{w}|}$, so $\frac{-2b}{2} = 3$ & $\boxed{b=3. \text{ So } a=1}$

$\therefore \nabla f|_P = \langle a,b \rangle = \langle 1,3 \rangle$.

2pts $Df|_{P,\vec{u}} = \nabla f|_P \cdot \frac{\vec{u}}{|\vec{u}|} = \frac{-1+6}{\sqrt{5}} = \sqrt{5}$.

**Q10 (8 points)** The dimensions of a closed rectangular box are measured to be 10 cm, 20 cm and 30 cm, and *each* measurement is correct to within 0.1 cm. Use differentials to estimate the largest possible error when the *total* surface area of the box is calculated from these measurements.

Let $x = \text{length} = 30$
$y = \text{width} = 20$
$z = \text{height} = 10$ 1pt

$S \overset{2pts}{=} 2xy + 2xz + 2yz$
$= 2(xy + xz + yz)$

$dS \overset{2pts}{=} 2(dx \cdot y + x \cdot dy + dx \cdot z + x \cdot dz + dy \cdot z + y \cdot dz)$
$= 2\left[(y+z)\,dx + (x+z)\,dy + (x+y)\,dz\right]$

$|dS| = 2\,|(y+z)\,dx + (x+z)\,dy + (x+y)\,dz|$
$\leq 2(y+z)|dx| + (x+z)|dy| + (x+y)|dz|$
$\leq 2\left((20+10)\frac{1}{10} + (30+10)\cdot\frac{1}{10} + (30+20)\cdot\frac{1}{10}\right)$
$= 2(3+4+5)$
$\boxed{= 24 \text{ cm}^2.}$ 1pt

2 pts