King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 201: First Major Exam, Summer 153 (120 minutes)

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Name: **Student ID:** **Serial Number:**

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Solve all problems. Show **full details** of your solution.

Question		Grade
1		/12
2		/18
3		/10
4		/6
5		/15
6		/11
7		/8
8		/10
9		/10
TOTAL		

Q1. Consider the parametric curve

$$x = -\sin t,\ y = \cos t,\ \frac{\pi}{4} \leq t \leq \frac{3\pi}{2}$$

(a) (4 points) Find a cartesian equation for the curve.

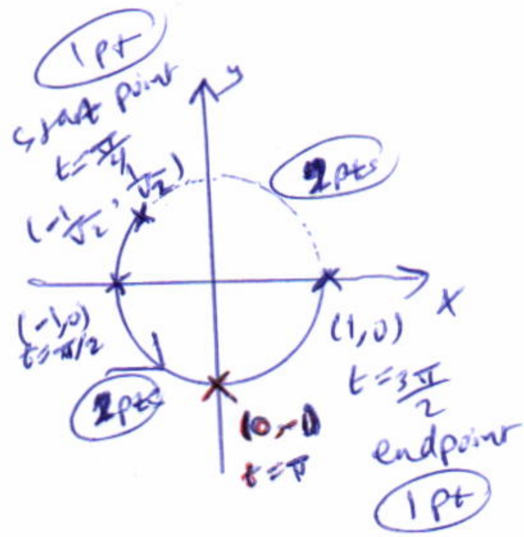
$$x^2 + y^2 = (-\sin t)^2 + (\cos t)^2$$
$$= \sin^2 t + \cos^2 t = 1$$

$$\boxed{x^2 + y^2 = 1}$$

(b) (8 points) Sketch the curve and indicate the direction in which the curve is traced as t increases (with an arrow) as well as the *start* and the *end* points.

2 pts

t	x	y
$\frac{\pi}{4}$	$\frac{-1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{2}$	-1	0
π	0	-1
$\frac{3\pi}{2}$	1	0



Q2. Consider the parametric curve C

$$x = t^2, y = t - t^3, t \in \mathbb{R}.$$

(a) **(8 points)** Find the equations of the tangents to the curve C at the point $(x, y) = (1, 0)$.

1pt $\frac{dx}{dt} = 2t$;

1pt $\frac{dy}{dt} = 1 - 3t^2$.

3 pts $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{2t}$

$x = 1$ 1pt

$t^2 = 1$

$t = \mp 1$

$y = 0$ ✓

$m\big|_{t=-1} = \frac{dy}{dx}\big|_{t=-1} = \frac{-2}{-2} = 1$

$y - 0 = 1(x-1)$ 1pt

$\boxed{y = x - 1}$

$m\big|_{t=1} = \frac{dy}{dx}\big|_{t=1} = \frac{-2}{2} = -1$ 1pt

$y - 0 = -1(x-1)$; $\boxed{y = 1 - x}$

(b) **(6 points)** Find the point(s) at which the tangent is horizontal.

⊛ horizontal tangent when $\frac{dy}{dt} = 0$ & $\frac{dx}{dt} \neq 0$. 2 pts

$\frac{dy}{dt} = 0$

$1 - 3t^2 = 0$; $t = \mp\frac{1}{\sqrt{3}}$. 1pt $\frac{dx}{dt}\big|_{\frac{1}{\sqrt{3}}} = \frac{2}{\sqrt{3}} \neq 0$ & $\frac{dx}{dt}\big|_{-\frac{1}{\sqrt{3}}} = \frac{-2}{\sqrt{3}} \neq 0$. 1pt

So, h. tangent at $\boxed{\left(\frac{1}{3}, \frac{2}{3\sqrt{3}}\right) \text{ \& } \left(\frac{1}{3}, \frac{-2}{3\sqrt{3}}\right)}$

1pt 1pt

(c) **(4 points)** Find the point(s) at which the tangent is vertical.

1pt C has a vertical tangent when $\frac{dx}{dt} = 0$ & $\frac{dy}{dt} \neq 0$

$\frac{dx}{dt} = 0$

$2t = 0$; $\boxed{t = 0}$ 1pt $\frac{dy}{dt}\big|_{t=0} = 1 \neq 0$. 1pt

So, C has a v. tangent ~~[illegible]~~ at $\boxed{(x, y) = (0, 0)}$.

Q3. (10 points) Find the length of the polar curve

$$r = 1 - \cos\theta$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad \text{1 pt}$$

$\frac{dr}{d\theta} = \sin\theta$ 1 pt

$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1-\cos\theta)^2 + (\sin\theta)^2$

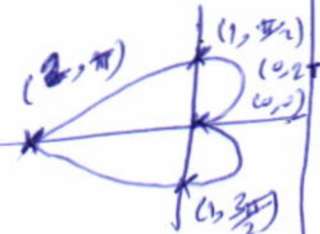
$= 1 - 2\cos\theta + (\cos^2\theta + \sin^2\theta)$

$= 2(1-\cos\theta)$ 2 pts

$= 2\left(1 - \cos 2\cdot\frac{\theta}{2}\right)$

$= 2\left(1 - \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)$

$= 4\sin^2\frac{\theta}{2}$ 2 pts



$= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta$

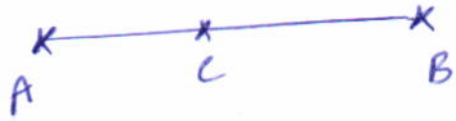
$= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta$ 1 pt

$= \int_0^{2\pi} 2\cdot\sin\frac{\theta}{2}\, d\theta$

$= \left(-4\cos\frac{\theta}{2}\right]_0^{\pi}$

$= -4(-1-1) = 8$ 3 pts

Q4 (6 points) Consider $A(1,2,5)$ and $B(0,1,3)$. Find the coordinates of the point C on the line segment AB such that $\left|\overrightarrow{AC}\right| = \frac{2}{3}\left|\overrightarrow{CB}\right|$.



Consider $C(a,b,c)$ on AB such that $|\overrightarrow{AC}| = \frac{2}{3}|\overrightarrow{CB}|$

As vectors $\boxed{\overrightarrow{AC} = \frac{2}{3}\overrightarrow{CB}}$ 2 pts

$\langle a-1, b-2, c-5\rangle = \frac{2}{3}\langle 0-a, 1-b, 3-c\rangle$

$$\boxed{\begin{aligned} a-1 &= -\frac{2}{3}a \\ b-2 &= \frac{2}{3}(1-b) \\ c-5 &= \frac{2}{3}(3-c) \end{aligned}} \quad \text{2 pts} \qquad \begin{aligned} \frac{5}{3}a &= 1 \\ \frac{5}{3}b &= \frac{8}{3} \\ \frac{5}{3}c &= 7 \end{aligned} \qquad \begin{aligned} a &= \frac{3}{5} \\ b &= \frac{8}{5} \\ c &= \frac{21}{5} \end{aligned}$$

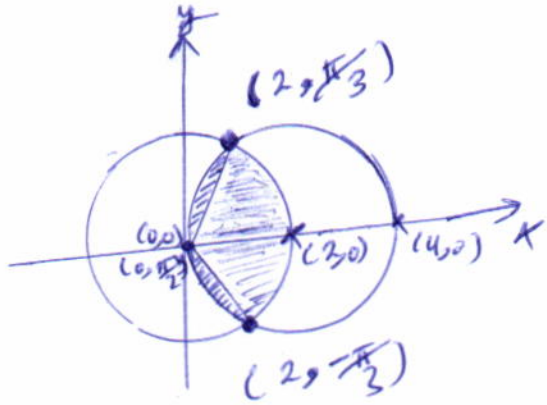
$C\left(\frac{3}{5}, \frac{8}{5}, \frac{21}{5}\right)$ 2 pts

Q5. (15 Points) Use polar coordinates to find the area of the region common to the circles

$$x^2 + y^2 = 4 \text{ and } x^2 + y^2 = 4x$$

$r^2 = 4$ $\quad$ $\boxed{r = 2}$

$r^2 = 4r\cos\theta$ $\quad$ $\boxed{r = 4\cos\theta}$



3 pts

$r_1 = r_2$

$2 = 4\cos\theta$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3} + 2n\pi$

or $\theta = -\frac{\pi}{3} + 2n\pi$

2 pts

$A = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$ $\quad$ 2 pts

$= 2(A_1 + A_2)$

$= 2\left(\boxed{\frac{1}{2}\int_0^{\pi/3} 2^2\, d\theta} + \boxed{\frac{1}{2}\int_{\pi/3}^{\pi/2} (4\cos\theta)^2\, d\theta}\right)$

2 pts $\quad$ 2 pts

A_1 $\quad$ A_2

1 pt

$\boxed{A_1 = \frac{2\pi}{3}}$

$A_2 = 8\int_{\pi/3}^{\pi/2} \cos^2\theta\, d\theta = 8\int_{\pi/3}^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta$ $\quad$ 2 pts

$= 4\left(\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_{\pi/3}^{\pi/2}\right) = 4\left(\frac{\pi}{2} + 0 - \frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$

$= 4\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$

$\boxed{= \frac{2\pi}{3} - \sqrt{3}}$

$A = 2(A_1 + A_2)$

$\therefore A = 2\left(\frac{2\pi}{3} + \frac{2\pi}{3} - \sqrt{3}\right)$

$\boxed{A = \frac{8\pi}{3} - 2\sqrt{3}.}$ $\quad$ 1 pt

Q6. Let $\vec{v} = <2, -3, 1>$ and $\vec{w} = <-1, 1, 4>$.

(a) (5 points) Find two *unit* vectors parallel to the vector $2\vec{v} - 3\vec{w}$.

$2\vec{v} - 3\vec{w} = 2<2,-3,1> - 3<-1,1,4>$

$= <7,-9,-10>$ 2pts ; $|2\vec{v}-3\vec{w}| = \sqrt{49+81+100}$

$= \sqrt{230}$ 1pt

1pt $\vec{u_1} = \frac{1}{\sqrt{230}}<7,-9,-10>$; $\vec{u_2} = -\vec{u_1} = \frac{1}{\sqrt{230}}<-7,9,10>$. 1pt

(b) (5 points) Find the vector projection of $\vec{v}$ on $\vec{w}$.

$proj_{\vec{w}} \vec{v} = \frac{\vec{v}\bullet\vec{w}}{\vec{w}\cdot\vec{w}} \vec{w}$ 2pts $= \frac{-2-3+4}{1+1+16} <-1,1,4>$ 1pt 1pt

$= \frac{-1}{18} <-1,1,4>$

$= <\frac{1}{18}, \frac{-1}{18}, -\frac{4}{18}>$. 1pt

Q7 (8 points) Let $\vec{r} = <x, y, z>$, $\vec{v} = <1, 2, 3>$ and $\vec{w} = <2, -1, 1>$. Show that the vector equation

$$(\vec{r} - \vec{v}) \bullet (\vec{r} - \vec{w}) = 0$$

represents a sphere. Find the center and the radius of that sphere.

$\vec{r} - \vec{v} = <x-1, y-2, z-3>$ 1pt

$\vec{r} - \vec{w} = <x-2, y+1, z-1>$ 1pt

$(\vec{r}-\vec{v}) \cdot (\vec{r}-\vec{w}) = 0$

$(x-1)(x-2) + (y-2)(y+1) + (z-3)(z-1) = 0$ 3pts

$x^2 - 3x + \cancel{2} + y^2 - y \cancel{-2} + z^2 - 4z + 3 = 0$.

$(x-\frac{3}{2})^2 + (y-\frac{1}{2})^2 + (z-2)^2 = \frac{9}{4} + \frac{1}{4} + 4 - 3$

$\boxed{(x-\frac{3}{2})^2 + (y-\frac{1}{2})^2 + (z-2)^2 = \frac{14}{4}}$ 1pt

Sphere with center $(\frac{3}{2}, \frac{1}{2}, 2)$ 1pt & radius $\frac{\sqrt{14}}{2}$. 1pt

Q8. (10 points) Check whether the four points $P(3,0,1)$, $Q(-1,2,5)$, $R(5,1,-1)$ and $S(0,4,2)$ lie on the same plane or not.

$\overrightarrow{PQ} = \langle -4, 2, 4 \rangle$ 1 pt

$\overrightarrow{PR} = \langle 2, 1, -2 \rangle$ 1 pt

$\overrightarrow{PS} = \langle -3, 4, 1 \rangle$ 1 pt

$$\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS}) = \begin{vmatrix} -4 & 2 & 4 \\ 2 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix}$$ 2 pts

2 pts

$$= -4(1-(-8)) - 2(2-6) + 4(8-(-3))$$

$$= -36 + 8 + 44$$

$$= 16 \neq 0$$ 1 pt

So, the four points are not in the same plane. 2 pts

Q9. (10 points) Check for symmetries and sketch the polar curve $r = |1 + 2\cos\theta|$.

Symmetries

x-axis ✓

$(r, -\theta)$: $r \overset{?}{=} |1 + 2\cos(-\theta)|$

$r \overset{\checkmark}{=} |1 + 2\cos\theta|$

Symmetric about the x-axis

1pt

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origin ✗

$(-r, \theta)$: $-r \overset{\times}{=} |1 + 2\cos\theta|$. 1pt

No symmetries about the origin or the y-axis.

1pt

$(r, \pi + \theta)$: $r = |1 + 2\cos(\pi + \theta)|$

$\overset{\times}{=} |1 - 2\cos\theta|$

1pt

$r \geq 0$ (absolute value).

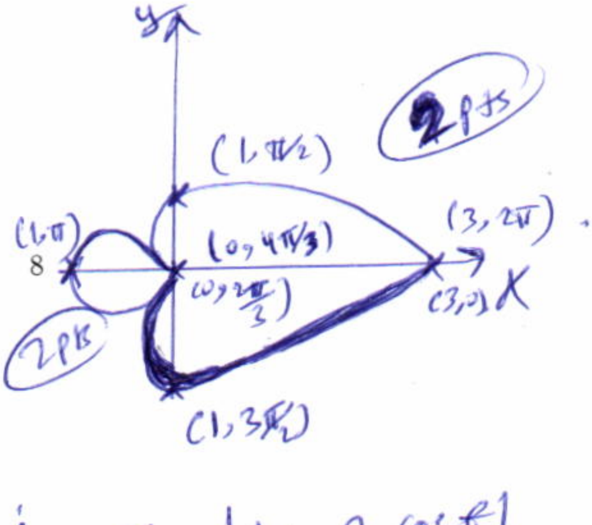
$r = 0$ iff $1 + 2\cos\theta = 0$

$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3} + 2n\pi$ } $n \in \mathbb{Z}$

or $\theta = \frac{4\pi}{3} + 2n\pi$

1pt



$-1 \leq \cos\theta \leq 1$

$-2 \leq 2\cos\theta \leq 2$

$-1 \leq 1 + \cos\theta \leq 3$;

$r = |1 + 2\cos\theta|$

$\boxed{1 \leq r \leq 3}$

θ	0	$\pi/2$	$2\pi/3$	π
r	3	1	0	1

1pts

We complete the graph using the fact that the curve is symmetric about the x-axis