"KEY"

King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 102 - Exam II - Term 153

Duration: 120 minutes

Name:________________ ID Number:________________

Section Number:________________ Serial Number:________________

Class Time: ________________ Instructor's Name:________________

Instructions:

1. Calculators, Smart devices and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 6 pages of problems (Total of 11 Problems)

Question Number	Points	Maximum Points
1		7
2		10
3		12
4		10
5		8
6		9
7		7
8		10
9		8
10		10
11		9
Total		100

1. **[7 points]** Find the integral $\int \frac{5x+1}{(2x+1)(x-1)}\,dx$.

0 2

$$\frac{5x+1}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$$

$$\Rightarrow 5x+1 = A(x-1) + B(2x+1)$$

Setting $x=1$ gives $B = \frac{6}{3} = 2$.

Setting $x=-\frac{1}{2}$ gives $-\frac{3}{2} = -\frac{3}{2}A \Rightarrow A = 1$ 0 2

Thus $\int \frac{5x+1}{(2x+1)(x+1)}\,dx = \int \frac{1}{2x+1}\,dx + \int \frac{2}{x-1}\,dx$

$= \frac{1}{2}\ln|2x+1| + 2\ln|x-1| + C$ 1 + 1 + 1

2. **[10 points]** Evaluate $\int_0^1 t\,e^{-t}\,dt$.

Let $u = t$ and $dv = e^{-t}\,dt$. 0 2

Then $du = dt$ " $v = -e^{-t}$ 0 2

$$\int_0^1 t e^{-t}\,dt = \left[-t e^{-t}\right]_0^1 + \int_0^1 e^{-t}\,dt$$ 0 2

$$= \left[-t e^{-t} - e^{-t}\right]_0^1$$ 0 1

$$= -e^{-1} - e^{-1} + 0 + e^{0}$$ 0 1 + 0 1

$$= 1 - \frac{2}{e}$$ 0 1

3. **[12 points]** Find $\displaystyle\int \frac{dx}{x^3\sqrt{9x^2-1}}$.

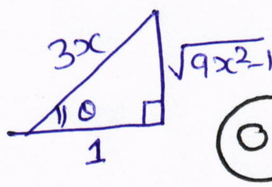
Let $3x = \sec\theta$. $\left(0 \le \theta < \frac{\pi}{2} \text{ or } \pi \le \theta < \frac{3\pi}{2}\right)$. Then $dx = \frac{1}{3}\sec\theta\tan\theta\,d\theta$ 02

$$\int \frac{dx}{x^3\sqrt{9x^2-1}} = \int \frac{1}{\frac{\sec^3\theta}{27}\cdot\tan\theta}\cdot\frac{1}{3}\sec\theta\tan\theta\,d\theta \quad 02$$

$$= 9\int \cos^2\theta\,d\theta \quad 01$$

$$= \frac{9}{2}\int (1+\cos 2\theta)\,d\theta \quad 01$$

$$= \frac{9}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C \quad 02$$



01

$$= \frac{9}{2}\left[\theta + \sin\theta\cos\theta\right] + C$$

$$= \frac{9}{2}\left[\sec^{-1}(3x) + \frac{\sqrt{9x^2-1}}{3x}\cdot\frac{1}{3x}\right] + C \quad 01 + 01 + 01$$

4. **[10 points]** Find the value of the integral $\displaystyle\int_0^{\pi/6} \sin^3\theta\,\sec^2\theta\,d\theta$.

$$= \int_0^{\pi/6} \frac{\sin^2\theta}{\cos^2\theta}\cdot\sin\theta\,d\theta \quad 01$$

$$= \int_0^{\pi/6} \frac{(1-\cos^2\theta)}{\cos^2\theta}\cdot\sin\theta\,d\theta \quad 01$$

Let $u = \cos\theta$. Then $du = -\sin\theta\,d\theta$ 02

$$\boxed{\begin{array}{l}\theta = 0 \Rightarrow u = 1\\ \theta = \frac{\pi}{6} \Rightarrow \frac{\sqrt{3}}{2}\end{array}} \quad 02$$

$$= -\int_1^{\sqrt{3}/2} \frac{(1-u^2)}{u^2}\,du \quad 01$$

$$= \int_1^{\sqrt{3}/2} (u^2-1)/u^2\,du$$

$$= \int_1^{\sqrt{3}/2} \left[1 - \frac{1}{u^2}\right]du = \left(u + \frac{1}{u}\right)_1^{\sqrt{3}/2} \quad 02$$

$$= \frac{\sqrt{3}}{2} + \frac{2}{\sqrt{3}} - 1 - 1 \quad 01$$

5. **[8 points]** Find the number $a < 0$ such that the average value of $f(x) = 3x^2 - 2x + 2$ on the interval $[a, 0]$ is equal to 8.

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx \qquad 01$$

$$8 = \frac{1}{0-a}\int_a^0 (3x^2-2x+2)\,dx \qquad 01$$

$$8 = \frac{1}{a}\int_0^a (3x^2-2x+2)\,dx$$

$$8a = \left[x^3 - x^2 + 2x\right]_0^a \qquad 02$$

or

$$8a = a^3 - a^2 + 2a$$

$$\Rightarrow a^3 - a^2 - 6a = 0$$

$$\Rightarrow a(a-3)(a+2) = 0 \qquad 03$$

$$\Rightarrow a = 0, \quad a = 3, \quad a = -2$$

$a = 0$ Rejected, $a = 3$ Rejected

So, $a = -2$ 01

6. [**9 points**] Use the substitution $t = \tan\frac{x}{2}$, $-\pi < x < \pi$ to find $\int \frac{dx}{\sin x - \cos x}$.

Let $t = \tan\frac{x}{2}$; $-\pi < x < \pi$. Then

$dx = \frac{2\,dt}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$ $\boxed{01+01+01}$

$$\int \frac{dx}{\sin x - \cos x} = \int \frac{\frac{2}{1+t^2}\,dt}{\frac{2t-1+t^2}{1+t^2}} \quad 01$$

$$= \int \frac{2}{t^2+2t-1}\,dt$$

$$= \int \frac{2}{(t+1)^2-(\sqrt{2})^2}\,dt \quad 01$$

$$= 2 \cdot \frac{1}{2\sqrt{2}} \ln\left|\frac{t+1-\sqrt{2}}{t+1+\sqrt{2}}\right| + C \quad 02$$

$$= \frac{1}{\sqrt{2}} \ln\left|\frac{\tan\frac{x}{2}+1-\sqrt{2}}{\tan\frac{x}{2}+1+\sqrt{2}}\right| + C \quad 01 + 01$$

7. [**7 points**] Find $\int \frac{x+1}{x^2+4x+6}\,dx$.

$$= \int \frac{x+1}{(x+2)^2+2}\,dx \quad 01$$

Let $x+2 = u$. Then $dx = du$ 01

$$= \int \frac{u-1}{u^2+2}\,du \quad 01$$

$$= \int \frac{u\,du}{u^2+2} - \int \frac{1}{u^2+2}\,du$$

$$= \frac{1}{2}\ln|u^2+2| - \frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}} + C \quad 02 + 01$$

$$= \frac{1}{2}\ln|(x+2)^2+2| - \frac{1}{\sqrt{2}}\tan^{-1}\frac{x+2}{\sqrt{2}} + C \quad 01$$

8. **[10 points]** Evaluate the improper integral $\int_{-1}^{0} \frac{e^{1/x}}{x^2}\, dx$ if possible.

$$\int_{-1}^{0} \frac{e^{1/x}}{x^2} dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{e^{1/x}}{x^2} dx$$ 01 + 01

Let $u = \frac{1}{x}$, Then $du = -\frac{dx}{x^2}$ 02

$$= \lim_{t\to 0^-} \int_{-1}^{\frac{1}{t}} -e^u\, du$$ 01

$$= \lim_{t\to 0^-} \left[-e^u\right]_{-1}^{\frac{1}{t}}$$ 01

$$= \lim_{t\to 0^-} \left[-e^{\frac{1}{t}} + e^{-1}\right]$$ 02

$$= 0 + \frac{1}{e}$$

$$= \frac{1}{e}$$ 02

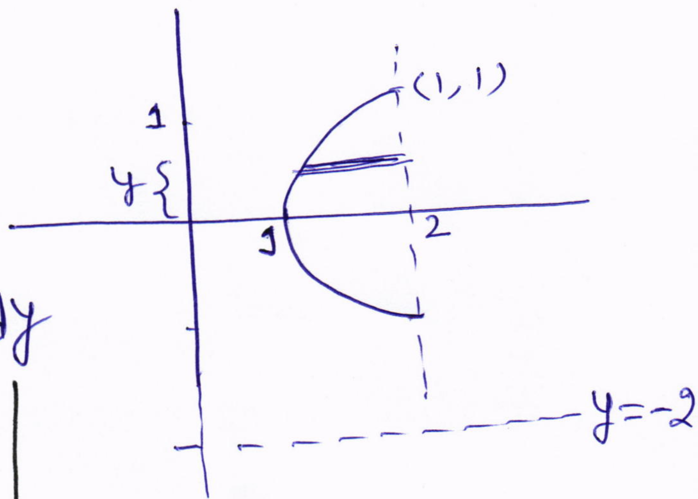
$\Rightarrow$ Improper integral converges to $\frac{1}{e}$.

9. **[8 points]** Use the **method of cylindrical shells** to find the volume generated by rotating the region bounded by the curves

$$x = y^2 + 1, \quad x = 2$$

about the line $y = -2$. **[Do not evaluate the integral]**

$$r = y - (-2)$$
$$= y + 2$$
$$H = 2 - (y^2 + 1)$$



$$V = 2\pi \int_{-1}^{1} (y+2)(2-(y^2+1))\, dy$$

01 02 02 02 01

10. **[10 points]** Find the integral $\int 2x \tan^{-1}(x)\,dx$.

By Parts:

$$u = \tan^{-1} x \quad ; \quad dv = 2x\,dx \qquad 02$$

$$du = \frac{1}{1+x^2}dx \qquad v = x^2 \qquad 02$$

$$\int 2x \tan^{-1} x\,dx = x^2 \tan^{-1} x - \int \frac{x^2}{1+x^2}dx \qquad 02$$

$$= x^2 \tan^{-1} x - \int \left[1 - \frac{1}{1+x^2}\right]dx \qquad 01$$

$$= x^2 \tan^{-1} x - x + \tan^{-1} x + C$$

01 01 01

11. **[9 points]** Find the area of the surface obtained by rotating the curve $x = \sqrt{4-y^2}$, $0 \le y \le 1$ about the y-axis.

$$\frac{dx}{dy} = \frac{1}{2\sqrt{4-y^2}}(-2y) = -\frac{y}{\sqrt{4-y^2}} \qquad 02$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2}{4-y^2} = \frac{4}{4-y^2}$$

$$S_y = \int_0^1 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \qquad 01 + 02$$

$$= \int_0^1 2\pi \sqrt{4-y^2} \cdot \frac{2}{\sqrt{4-y^2}}\,dy \qquad 1 + 1$$

$$= 4\pi \int_0^1 dy \qquad 01$$

$$= 4\pi. \qquad 01$$