KEY

King Fahd University of Petroleum & Minerals

Department of Mathematics & Statistics

MATH 101-Exam I-Term 153

Duration: 90 minutes

Name: KEY ID:

Section Number: Serial Number:

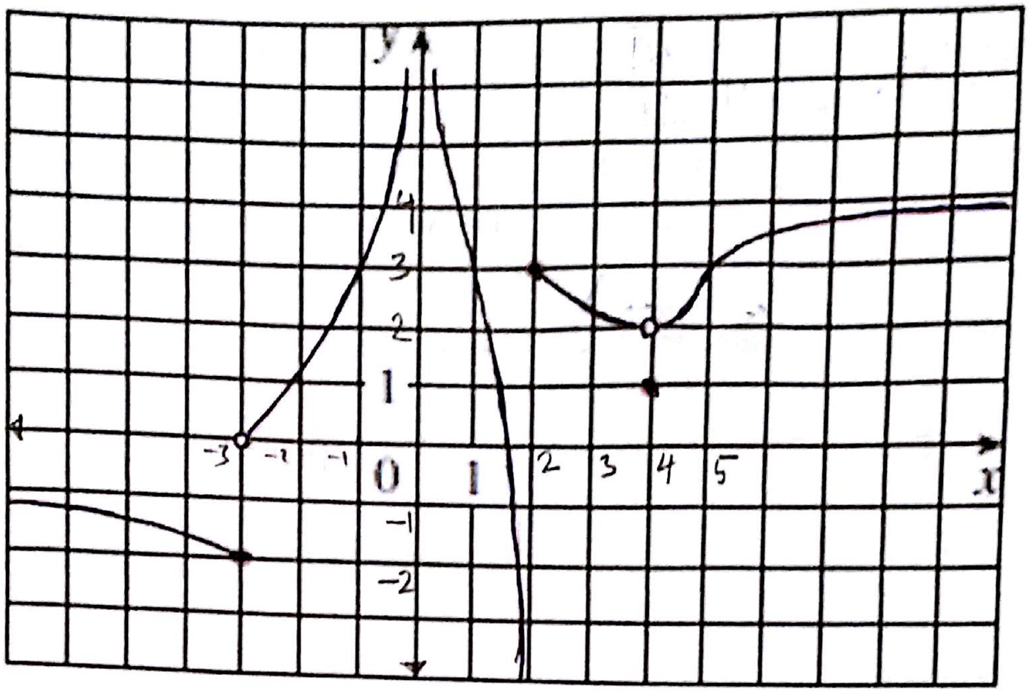
Class Time: Instructor's Name:

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure you have 6 pages of problems (Total of 8 Problems).

Page Number	Full Mark	Marks Obtained
One	9	
Two	29	
Three	17	
Four	15	
Five	21	
Six	9	
Total	**100**	

Question 1 (5+2+2+12 +10 =32 points)



Using the graph of the function $f(x)$ above:

a) Find each limit, or explain why it does not exist.

1 i. $\lim_{x\to -3^+} f(x) = 0$

2 ii. $\lim_{x\to 0} f(x) = +\infty$

$\lim_{x\to 0^+} f(x) = +\infty = \lim_{x\to 0^-} f(x)$

1 iii. $\lim_{x\to -\infty} f(x) = -1$

1 iv. $\lim_{x\to \infty} f(x) = 4$

b) State the equation(s) of the vertical asymptote(s) if exist.

1 $\boxed{x=0}$ is a V. A. from part (ii)

1 $\boxed{x=2}$ " " " " : $\lim_{x\to 2^-} f(x) = -\infty$

c) State the equation(s) of the horizontal asymptote(s) if exist.

1 $\boxed{y=-1}$ is a H. A. from part (iii)

1 $\boxed{y=4}$ " " " " (iv)

d) At what numbers is f discontinuous? State the type of discontinuity. Explain.

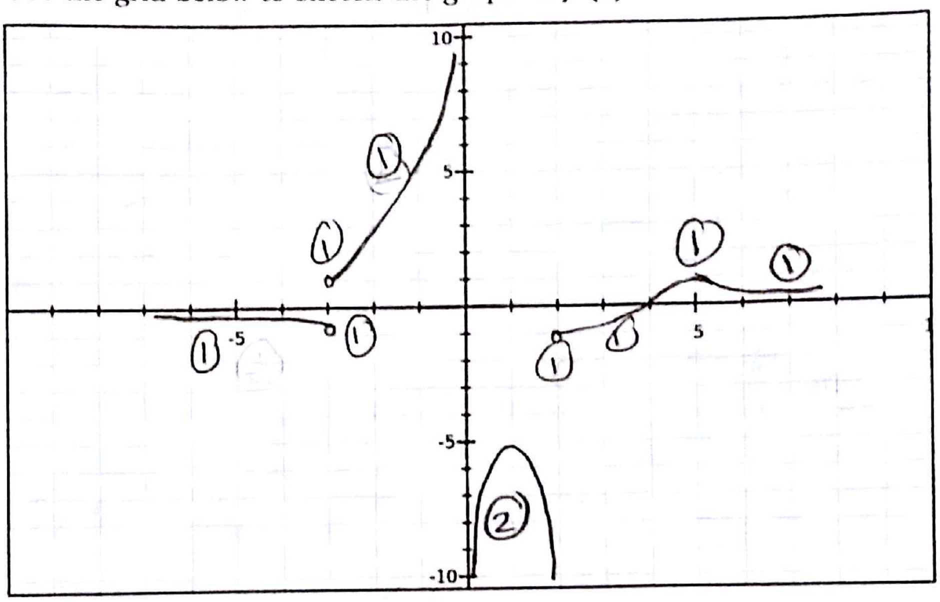
1 $\boxed{x=-3}$ is a Jump 1 discont. $\{\lim_{x\to-3^-} f(x) = -2 \neq 0 = \lim_{x\to-3^+} f(x)\}$ 1

1 $\boxed{x=0}$ is an Infinite 1 " { from part (ii) } 1

1 $\boxed{x=2}$ is " " 1 " $\{\lim_{x\to2^-} f(x) = -\infty\}$ 1

1 $\boxed{x=4}$ is a removable 1 " $\{\lim_{x\to4} f(x) = 2 \neq 1 = f(4)\}$ 1

e) Use the grid below to sketch the graph of $f'(x)$.



Question 2 (1+3+3=7 points)

Use limits to find all vertical asymptotes, if any exists, of the function $f(x) = \frac{\ln x}{x-2}$.

$* \ \text{dom}(f) = (0,2)\cup(2,+\infty)$

$* \ \lim_{x\to0^+} f(x) = \lim_{x\to0^+} \frac{\ln x}{x-2} = +\infty$ 1 $\Rightarrow \boxed{x=0}$ 1 is V.A. 1

$* \ \lim_{x\to2^-} f(x) = \lim_{x\to2^-} \frac{\ln x}{x-2} = -\infty$ 1 $\Rightarrow \boxed{x=2}$ 1 is V.A. 1

and $\lim_{x\to2^+} f(x) = +\infty$ 1

Question 3 (2+5+6+4=17 points)

Find the limit if it exists:

a) $\lim\limits_{x\to-3} \frac{3-|x|}{3+x} = \lim\limits_{x\to-3} \frac{3+x}{3+x} = \lim\limits_{x\to-3} 1 = \boxed{1}$ (1) (1)

b) $\lim\limits_{x\to0^+} (2x+x^2)\cos\frac{1}{x^2} = 0$ (1) (Using Squeeze Theorem) (1)

$-1 \le \cos\frac{1}{x^2} \le 1$ (1)

$-(2x+x^2) \le (2x+x^2)\cos\frac{1}{x^2} \le (2x+x^2)$ (1)

$\underbrace{-\lim\limits_{x\to0^+}(2x+x^2)}_{=0} \le \lim\limits_{x\to0^+}(2x+x^2)\cos\frac{1}{x^2} \le \underbrace{\lim\limits_{x\to0^+}(2x+x^2)}_{=0}$ (1)

c) $\lim\limits_{x\to-\infty} \frac{10-7x^5}{\sqrt{x^{10}-x^4}} = \lim\limits_{x\to-\infty} \frac{10-7x^5}{|x^5|\sqrt{1-\frac{x^4}{x^{10}}}}$ (2)

$= \lim\limits_{x\to-\infty} \frac{10-7x^5}{-x^5\sqrt{1-\frac{1}{x^6}}}$ (1) $= \lim\limits_{x\to-\infty} \frac{x^5\left(\frac{10}{x^5}-7\right)}{-x^5\sqrt{1-\frac{1}{x^6}}}$ (1)

$= \lim\limits_{x\to-\infty} \frac{\frac{10}{x^5}-7}{-\sqrt{1-\frac{1}{x^6}}}$ (1) $= \frac{-7}{-1} = \boxed{7}$ (1)

d) $\lim\limits_{x\to-\frac{3}{2}} \frac{2x^2+x-3}{2x^2+13x+15} = \frac{0}{0}$

Factorize: $2x^2+x-3 = (2x+3)(x-1)$ (1)

and $2x^2+13x+15 = (2x+3)(x+5)$ (1)

$\Longrightarrow$

$\lim\limits_{x\to-\frac{3}{2}} \frac{(2x+3)(x-1)}{(2x+3)(x+5)} =$

$= \lim\limits_{x\to-\frac{3}{2}} \frac{x-1}{x+5}$ (1) $= \frac{-\frac{5}{2}}{\frac{7}{2}}$

$= \boxed{\frac{-5}{7}}$ (1)

Question 4 (**5**+1=**6** points)

a) Use the ε-δ definition of the limit to show that $\lim_{x\to 2}(4x-9) = -1$.

$f(x) = 4x - 9$, $a = 2$, $L = -1$

(1) $\forall \epsilon > 0, \exists \delta > 0$, such that

(1) $|f(x) - L| < \epsilon$ if $0 < |x - a| < \delta$

(1) $|4x - 9 + 1| < \epsilon$ if $0 < |x - 2| < \delta$

$|4x - 8| < \epsilon$ if . . .

$4|x - 2| < \epsilon$ if . . .

(1) $|x - 2| < \frac{\epsilon}{4}$ if . . .

(1) $\Rightarrow \boxed{\delta \le \frac{\epsilon}{4}}$

b) In part (a), find values of δ that correspond to $\varepsilon = 0.1$.

(1) Since $\delta \le \frac{\epsilon}{4} \Rightarrow \delta \le \frac{0.1}{4} = \boxed{0.025}$

Question 5 (**8** points)

For what values of k the function $f(x)$ has removable discontinuity at $x = 0$?

$$f(x) = \begin{cases} k - \sin x & if \quad x < 0 \\ 1 & if \quad x = 0 \\ x^2 - k^2 + 2 & if \quad x > 0 \end{cases}$$

(1) If $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) \neq f(0) \Rightarrow$

f has removable discont. at $(x = 0)$

(1) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(k - \sin x) = k$

(1) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x^2 - k^2 + 2) = 2 - k^2$

$\Rightarrow k^2 + k - 2 = 0 \Rightarrow \boxed{k = -2}$ (1) or $\boxed{k = +1}$ (1)

* For $k = 1$ $\lim_{x\to 0} f(x) = \lim_{x\to 0^-}(1 - \sin x) = 1 = f(0) \Rightarrow$ (1)

f is cont. at $x = 0$ which is rejected. (1)

* For $k = -2$ $\lim_{x\to 0} f(x) = \lim_{x\to 0^-}(-2 - \sin x) = -2 \neq 1 = f(0) \Rightarrow$ (1)

f has removable discontinuity at $(x = 0)$ (1)

Question 6 (7+5=12 points)

Let f be the function defined by: $f(x) = x^3 \cos\frac{1}{x}$ if $x \neq 0$, $f(0) = 0$.

a) Is the function f continuous at 0? Explain.

(1) Yes, $f(x)$ is continuous at $\boxed{x=0}$, because:

$\lim_{x\to 0} f(x) = 0 = f(0)$ (using Squeeze Theorem) (1)

(1) $-1 \leq \cos\frac{1}{x} \leq 1$

(1) $-x^3 \leq x^3\cos\frac{1}{x} \leq x^3$

(1) $\lim_{x\to 0^-}(-x^3) \leq \lim_{x\to 0^-} x^3\cos\frac{1}{x} \leq \lim_{x\to 0^-} x^3$

(1) $0 \geq \lim_{x\to 0^-} x^3\cos\frac{1}{x} \geq 0$

$-1 \leq \cos\frac{1}{x} \leq 1$

$-x^3 \leq x^3\cos\frac{1}{x} \leq x^3$

$\lim_{x\to 0^+}(-x^3) \leq \lim_{x\to 0^+} x^3\cos\frac{1}{x} \leq \lim_{x\to 0^+} x^3$ (1)

$0 \leq \lim_{x\to 0^+} x^3\cos\frac{1}{x} \leq 0$

b) Find all the asymptotes of the function f, if any exists. Explain.

* Since $f(x)$ is cont. everywhere (1) then $f(x)$ has $\boxed{\text{NO V.A.}}$ (1)

* $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x^3\cos\frac{1}{x} = +\infty$ (1)
* $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^3\cos\frac{1}{x} = -\infty$ (1)

$\Rightarrow f(x)$ has $\boxed{\text{NO H.A.}}$ (1)

Question 7 ([illegible] points)

Show that the equation $x^3 - 2x^2 + 3x = 5$ has at least one solution. Show all your work and name any theorem you use.

* Let $f(x) = x^3 - 2x^2 + 3x - 5$ $\Rightarrow$ (1)

$f(x)$ is continuous ~~everywhere~~, because it is a polynomial. (1)

* $f(1) = 1 - 2 + 3 - 5 = -3 < 0$ (1)

$f(2) = 8 - 8 + 6 - 5 = +1 > 0$ (1) $\Rightarrow$

$f(x)$ is cont. on $[1, 2]$ (1)

* Then by (1) IVT, $\exists$ at least one $c \in (1, 2)$ (1) such that

(1) $f(c) = 0 \Rightarrow \boxed{c^3 - 2c^2 + 3c = 5}$

(1)

21

Question 8 (6+3=9 points)

a) Find the instantaneous rate of change of the function $y = \frac{1}{\sqrt{x-1}}$ at $x = 5$.

$$1 \quad f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h-1}} - \frac{1}{\sqrt{x-1}}}{h} = \frac{0}{0}$$

$$1 \quad = \lim_{h\to 0} \frac{\sqrt{x-1} - \sqrt{x+h-1}}{h\sqrt{x-1}\sqrt{x+h-1}} \cdot \frac{\sqrt{x-1} + \sqrt{x+h-1}}{\sqrt{x-1} + \sqrt{x+h-1}}$$

$$1 \quad = \lim_{h\to 0} \frac{x-1-(x+h-1)}{h\sqrt{x-1}\sqrt{x+h-1}(\sqrt{x-1}+\sqrt{x+h-1})}$$

$$= \lim_{h\to 0} \frac{-h}{h\sqrt{x-1}\sqrt{x+h-1}(\sqrt{x-1}+\sqrt{x+h-1})}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{x-1}\sqrt{x+h-1}(\sqrt{x-1}+\sqrt{x+h-1})}$$

$$1 \quad = \frac{-1}{2(x-1)\sqrt{x-1}} = \frac{-1}{2(x-1)^{3/2}}$$

$$\Rightarrow f'(5) = \frac{-1}{1\ 2\cdot 4^{3/2}} = \boxed{\frac{-1}{16}} \quad 1$$

b) Find the equation of the tangent line to the curve of y at $x = 5$.

From part (a) $\Rightarrow m = f'(5) = -\frac{1}{16}$ 1

$\Rightarrow y - y_1 = m(x - x_1)$

1 $y - \frac{1}{2} = -\frac{1}{16}(x-5)$

$y = -\frac{1}{16}x + \frac{5}{16} + \frac{1}{2} \Rightarrow$

$y = -\frac{1}{16}x + \frac{13}{16}$ 1

is the equation of the tangent line