King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

Math 551 Abstract Algebra - Term 152
Final Exam (Duration = 6 hours)

This is a closed book closed notes exam

---

(1) [14 points] (a) Show that a summand of an injective module is injective.
(b) Let $0 \to E_1 \to E_2 \to E_3 \to 0$ be an exact sequence of modules with E_1 being injective. Show that E_2 is injective if and only if E_3 is injective.

---

(2) [14 points] Let (A, $\mathbf{m}$) be a local ring, M a finitely generated A-module, and N an A-submodule of M. Prove:
(a) If $M = N + \mathbf{m}M$, then $M = N$.
(b) $M/\mathbf{m}M = \langle x_1+\mathbf{m}M, \ldots, x_n+\mathbf{m}M \rangle$ if and only if $M = \langle x_1, \ldots, x_n \rangle$.

---

(3) [14 points] Let A be a commutative ring.
(a) Let I be an ideal of A and M an A-module with IM = 0. Prove that M is Noetherian (resp., Artinian) as an A-module if and only if M is Noetherian (resp., Artinian) as an (A/I)-module.
(b) Let V be a vector space. Prove: $\dim(V) < \infty \Leftrightarrow$ V is Noetherian $\Leftrightarrow$ V is Artinian.

---

(4) [14 points] Recall that a ring A is Artinian if it is Artinian as an A-module; i.e., A satisfies dcc on its ideals or, equivalently, every non-empty set of ideals (of A) has a minimal element. Let A be an Artinian commutative ring. Prove:
(a) All prime ideals of A are maximal, and then Nil(A) = J(A).
(b) There is only a finite number of maximal ideals.
(c) J(A) is nilpotent.

---

(5) [14 points] Let R be a ring. Recall that an R-module E is semisimple if E is a sum of simple modules or, equivalently, if every submodule of E is a summand. Let E be a semisimple module.
(a) Prove that every nonzero submodule of E contains a simple module.
(b) Let F be a nonzero submodule of E. Prove that F and E/F are semisimple.

---

(6) [15 points] Recall that a ring R is semisimple if it is semisimple as an R-module. Prove that the following conditions are equivalent for a ring R:
(i) R is semisimple;
(ii) Every (left) R-module is semisimple;
(iii) Every (left) R-module is injective;
(iv) Every (left) R-module is projective.

---

(7) [15 points] Let R be a semisimple ring with $I_1, I_2, \ldots, I_s$ its non-isomorphic simple (left) ideals. Let E be a nonzero R-module. Prove that $E = \bigoplus_{1 \leq i \leq s} E_i$ where E_i = Sum of all simple submodules of E isomorphic to I_i, for i = 1, . . ., s.