King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

Math 551 Abstract Algebra - Term 152
Mid-term Exam (Duration = 6 hours)

This is a closed book closed notes exam

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(1) [10 points] Let $\{R_i\}_{i\in\Delta}$ be a family of rings and consider the ring $R:=\prod_{i\in\Delta} R_i$. Prove that if Δ is finite, then every ideal of R has the form $\prod_{i\in\Delta} I_i$ for some ideals I_i of R_i ($i\in\Delta$). Is this fact true if Δ is infinite? (Justify)

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(2) [10 points] Let R be a ring. Prove:

$$0 \to Y' \to Y \to Y'' \text{ exact} \Leftrightarrow 0 \to \mathrm{Hom}_R(X, Y') \to \mathrm{Hom}_R(X, Y) \to \mathrm{Hom}_R(X, Y'') \text{ exact}, \ \forall\ X$$

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(3) [10 points] Let R be a commutative ring and let M be an R-module. Prove the implications:

$$M \text{ free} \Rightarrow M \text{ projective} \Rightarrow M \text{ flat}$$

and provide explicit counterexamples for the converses.

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(4) [10 points] Prove that if a short exact sequence of modules $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ splits, then:

- **(a) [4 points]** $M = \mathrm{Ker}(g) \oplus \mathrm{Im}(\varphi)$, where $g\circ\varphi = 1_{M''}$
- **(b) [4 points]** $M = \mathrm{Ker}(\psi) \oplus \mathrm{Im}(f)$, where $\psi\circ f = 1_{M'}$
- **(c) [2 points]** $M \cong M' \oplus M''$

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(5) [15 points] Let R be an integral domain containing a field k. Prove that if R is a finite dimensional vector space over k, then R is a field.

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(6) [20 points] Let R be a commutative ring. An R-module M is called <u>faithfully flat</u> if M is flat and $M\otimes N = 0 \Rightarrow N = 0$. Prove that the following conditions are equivalent:

- **(i)** M is faithfully flat;
- **(j)** M is flat and if $u: E \to F$ is an R-map with $u \neq 0$, then $T_M(u): M\otimes E \to M\otimes F$ is also $\neq 0$;
- **(k)** M is flat and $pM \neq M$ for all <u>maximal</u> ideals p of R;
- **(l)** $N' \to N \to N''$ is exact $\Leftrightarrow M\otimes N' \to M\otimes N \to M\otimes N''$ is exact.

[Each implication = **5 points]**

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(7) [25 points] Let R be a commutative ring and E, F two R-modules. We say that E is F-flat if:

$$0 \to F' \to F \text{ exact} \Rightarrow 0 \to E\otimes F' \to E\otimes F \text{ exact}$$

- **(a) [5 points]** Prove: E is F-flat $\Rightarrow$ E is G-flat for every submodule G of F. [Use Snake Lemma]
- **(b) [5 points]** Prove: E is F-flat $\Rightarrow$ E is (F/G)-flat for every submodule G of F. [Use Snake Lemma]
- **(c) [5 points]** Assume $F = F_1\oplus F_2$. Prove: E is F_i-flat (i=1, 2) $\Leftrightarrow$ E is F-flat. [Use Snake Lemma]
- **(d) [5 points]** Assume $F = \oplus_{i\in\Delta} F_i$. Prove: E is F_i-flat ($\forall\ i\in\Delta$) $\Leftrightarrow$ E is F-flat. [Use (c)]
- **(e) [5 points]** Prove: E is flat $\Leftrightarrow$ E is R-flat. [Use (d) and (b)]