King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 550: Linear Algebra

Final Exam, Spring Semester 152 (180 minutes)

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Part I. Solve *any* two of Q1, Q2 and Q3:

Q1. (20 points) Let T be a linear operator on the finite dimensional inner product space V. Show that:

(a) There exists a unique linear operator T^* on V such that

$$< T(\alpha), \gamma >=< \alpha, T^*(\gamma) >$$

for all α, $\gamma \in V$.

(b) In any orthonormal basis $\mathcal{B}$ of V, we have

$$[T^*]_{\mathcal{B}} = ([T]_{\mathcal{B}})^*.$$

Q2. (20 points) Let T be a linear operator on the finite dimensional inner product space V. Show that:

(a) If W is a subspace of V which is invariant under T, then $W^{\perp}$ is invariant under T^*.

(b) If T is self-adjoint, then there exists an orthonormal basis $\mathcal{B}$ of V consisting of characteristic vectors of T.

Q3. (20 points) Let V be a finite dimensional vector space over $\mathbb{R}$. Show that:

(a) if f is a symmetric bilinear form on V, then there exists an orthonormal basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is diagonal.

(b) $rank(L_f) = rank(R_f)$, where

$$L_f, R_f : V \longrightarrow V^*,\ (L_f(\alpha))(\gamma) =< \alpha, \gamma >= (R_f(\gamma))(\alpha)$$

for all α, $\gamma \in V$.

Part II. Solve *all* of the following questions:

Q4. (15 points) Let $T : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ be a linear operator represented (w.r.t. the standard basis) by the matrix

$$A = \begin{bmatrix} 3 & 4 & 0 \\ -1 & -1 & 0 \\ 3 & 9 & 2 \end{bmatrix}.$$

(a) Find the Jordan canonical form J of A.
(b) find an (invertible) matrix P such that $P^{-1}AP = J$.

Q5. (25 points) Let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$.

(a) Let

$$B = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and consider

$$W = \{A \in V \mid \; AB = 0\}.$$

Find $f(B)$, where $f \in W^{\perp}$, $f(I) = 0$ and $f(C) = 3$.

(b) Let $W = \mathbb{R}^{2\times 1}$. Define for any $A \in V$, the map

$$f_A : W \times W \longrightarrow \mathbb{R}, \; (X, Y) \longmapsto Y^t AX.$$

Show that (W, f_A) is an inner product space if and only if A is symmetric, $A_{11} > 0$, $A_{22} > 0$ and $\det(A) > 0$.

Q6. (20 points) Prove or disprove:

(1) All finite dimensional inner product spaces are isomorphic as inner product spaces.

(2) If $T : V \longrightarrow V$ is a linear operator with a left inverse S (i.e. $S \circ T = I$), then T is invertible.

(3) If V is an inner product space and $W \leq V$ is a subspace, then $W = (W^{\perp})^{\perp}$.

(4) Let V be a vector space, U, W be subspaces of V such that $V = U \oplus W$. If T is a linear operator on V such that U is invariant under T, then W is also invariant under T.

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