King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 550: Linear Algebra

First Exam, Spring Semester 152 (120 minutes)

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Q1. (20 points) Let V be a vector space, and W_1, W_2 be two subspaces of V.

(a) Show that if W_1 and W_2 are finite dimensional, then

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

(b) Show that if $W_1 + W_2 = V$ and $W_1 \cap W_2 = 0$, then for every $\alpha \in V$ there exist *unique* $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$ such that $\alpha = \alpha_1 + \alpha_2$.

Q2. (20 points) Let V be a finite dimensional vector space.

(a) Show that if $W \lneqq_F V$ is a proper subspace of V and $\alpha \in V \backslash W$, then there exists $f \in V^*$ such that $f(W) = 0$ but $f(\alpha) \neq 0$.

(b) Show that if W_1, W_2 are subspaces of V and $W_1^0 = W_2^0$, then $W_1 = W_2$.

Q3. (15 points) Let $\alpha_1 = (1, 1, 0)$, $\alpha_2 = (1, 0, 1)$ and $\alpha_3 = (0, 1, 1)$.

(a) Show that $\beta = \{\alpha_1, \alpha_2, \alpha_3\}$ is a basis of $\mathbb{R}^3$.

(b) Let $\alpha = (a, b, c) \in \mathbb{R}^3$. Find $[\alpha]_\beta$ (the coordinate matrix of α w.r.t. the *ordered* basis β).

Q4. (10 points) Let θ be a fixed angle. Show that the linear transformation

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ (x, y, z) \longmapsto (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z)$$

is invertible and find its inverse.

Q5. (20 points) Consider the linear transformation

$$T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3,\ (x, y, z, w) \longmapsto (z-y+z+w, 2x-2y+3z+4w, 3x-3y+4z+5w).$$

(a) Find the rank and the nullity of T.

(b) Find a basis for $\mathrm{Im}(T)$.

(c) Find a basis for $\mathrm{Null}(T)$.

Q6. (15 points) Prove or disprove:

(a) If V is a finite dimensional vector space, then any $0 \neq S \subsetneqq V$ is linearly independent.

(b) If V is an n-dimensional vector space and $T : V \longrightarrow V$ is a linear operator such that $\mathrm{Null}(T) = T(V)$, then n is even.

(c) If V and W are finite dimensional vector spaces and $T : V \longrightarrow W$ is a 1-1 linear transformation, then T is invertible.

GOOD LUCK