King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
MATH 531 [Real Analysis]
Semester (152)

Exam II: May 10, 2016 **Time allowed: 2hrs**

(Q1) Let f be a real-valued function defined on a measurable domain D. Define measurability of the function f. Show that if f is measurable, then f^2 is a measurable function.

(Q2) Define $f : \mathbb{R} \to \{0, 1\}$ by

$$f(x) = \begin{cases} 1 & \text{if } \ x \in Q' \ \text{(set of irrationals)} \\ 0 & \text{if } \ x \in Q \ \text{(set of rationals)} \end{cases}$$

Use an appropriate result to check whether or not the function f is measurable.

(Q3) Show that a continuous function defined on a measurable domain D is measurable.

(Q4) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function and $g : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. Then prove that $g\,o\,f$ is a measurable function.

(Q5) (a) Find a measurable set that is not Borel set.

(b) Show that an extended real-valued function f defined on a set D of measure zero need not be Borel measurable.

(Q6) Let f and g non-negative measurable functions defined on a measurable set E. Define $\int_E f\,dm$ where m stands for Lebesgue measure.

Hence show that $\int_E (f+g)\,dm = \int_E f\,dm + \int_E g\,dm$

(Q7) (a) State and prove Montone Convergence Theorem (MCT).

(b) Check whether or not (MCT) holds for the sequence of functions: $f_n : \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = \frac{1}{n} \mathcal{X}_{[n,\infty)}$.

(Q8) Let g be integrable and $\{f_n\}$ a sequence of measurable functions on E such that $|f_n| \leq g$. If $f_n \to f$ a.e. on E, then $\int_E f = \lim_n \int_E f_n$.
Explain why and how this result does not hold for $E = (0, 1]$ and

$$f_n(x) = \begin{cases} n & \text{if } x \in \left(0, \dfrac{1}{n}\right] \\ 0 & \text{if } x \in \left(\dfrac{1}{n}, 1\right] \end{cases}$$

(Q9) (a) Let $\{u_n\}$ be a sequence of non-negative measurable functions and $f = \sum_{n=1}^{\infty} u_n$. Then show that

$$\int f = \sum_{n=1}^{\infty} \int u_n$$

(b) Give an example of a function f such that $|f|$ is measurable but f is not measurable.