King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 460
Final Exam, Semester II, 2015-2016
Net Time Allowed: 180 minutes

Name:————————————————————
ID:——————————Section:——————Serial:——————

Q#	Marks	Maximum Marks
1		10
2		25
3		13
4		20
5		10
6		12
7		10
8		5
9		10(bonus)
10		10(bonus)
Total		130

1. Write clearly.
2. Show all your steps.
3. No credit will be given to wrong steps.
4. Do not do messy work.
5. Calculators and mobile phones are NOT allowed in this exam.
6. Turn off your mobile.

1. Cooperating Species. Consider two species that survive in a symbiotic relationship in the sense that the population of each species decreases at a rate equal to its existing number but increases at a rate equal to the existing number in the other population.

   (a) If there are initially 200 of species I and 400 of species II, determine the number of each species at all future times.

   (b) Discuss the long-run behavior of each species.

2. For $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$. Find the following

(a) The SVD for A

(b) Rank(A)

(c) Orthonormal bases for R(A) and N(A^T)

(d) Compute the orthogonal projectors onto each of the four fundamental subspaces associated with A

(e) Use part(a) to calculate the Moore-Penrose inverse of A (i.e. $A^{\dagger}$)

(f) Find the minimum length least squares solution of the equation $Ax = b$

(g) The condotional number for A

3. For $A \in \mathbb{R}^{m\times n}$ with $p = \min\{m, n\}$, let $\{\sigma_1, \sigma_2, \cdots, \sigma_p\}$ and $\{\beta_1, \beta_2, \cdots, \beta_p\}$ be all singular values (nonzero as well as any zero ones) for A and $A + E$, repectively.

   Prove that

$$|\sigma_k - \beta_k| \leq \|E\|_2, \qquad \text{for each} \ \ k = 1, 2, \cdots, p.$$

4. Let $A = \begin{bmatrix} 6 & 2 & 8 \\ -2 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix}$

(a) Calculate the algebric and geometric multiplicity for each eigenvalue of the matrix A

(b) Find the Jordan form of A

(c) Find the spectral projectors for A

(d) Find $\sin(A)$

5. Consider the quadratic polynomial

$$p(x) = 9x_1^2 + 7x_2^2 + 3x_3^2 - 2x_1x_2 + 4x_1x_3 - 6x_2x_3 + 6x_2 - 7x_3 + 5$$

(a) Write $p(x)$ as $x^TKx - 2x^Tf + c$, where K is a 3×3 symmetric matrix and x is a 3×1 column vector.

(b) Show that K is positive definite

(c) Find the minimum value of $p(x)$

6. The following steps will consider the perturbation linear system

   (a) If $\lim_{n\to\infty} A^n = 0$, prove that $(I - A)$ is nonsingular and $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$

   (b) Using (a) and first-order approximation, show that

   $$(A + B)^{-1} \approx A^{-1} - A^{-1}BA^{-1}$$

   (c) If a nonsingular linear system $Ax = b$ is sligtly perturbed to $(A + B)\widetilde{x} = b$, show that

   $$\frac{\|x - \widetilde{x}\|}{\|x\|} \leq \kappa(A) \left\{ \frac{\|B\|}{\|A\|} \right\}$$

7. Use Rayleigh Quotient to approximate the dominant eigenvalue and the corresponding eigenvectors for $A = \begin{bmatrix} 7 & -4 & 2 \\ 16 & -9 & 6 \\ 8 & -4 & 5 \end{bmatrix}$

8. Given $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$ are normal matrices, show that $A \otimes B$ is also normal.

9. Solve Sylvester equation $AX + XB = C$, where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 3 & 0 \\ 1 & -1 \end{bmatrix}$$

10. Find the Fourier matrix of order 4 and its inverse, then calculate the following

(a) The discrete Fourier transform of $\begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}$

(b) The inverse Fourier transform of $\begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}$