King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 460
Major Exam II, Semester II, 2015-2016
Net Time Allowed: 120 minutes

Name:____
ID:____________________Section:__________Serial:______________

Q#	Marks	Maximum Marks
1		10
2		12
3		12
4		10
5		10
6		10
7		10
8		10
Total		84

1. Write clearly.
2. Show all your steps.
3. No credit will be given to wrong steps.
4. Do not do messy work.
5. Calculators and mobile phones are NOT allowed in this exam.
6. Turn off your mobile.

1. With respect to the inner product for matrices given by $\langle A, B\rangle = \text{Trace}(A^T B)$, verify that the set

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right\}$$

is an orthonormal basis for $\mathcal{R}^{2\times 2}$, and then compute the Fourier expansion of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ with respect to $\mathcal{B}$.

2. Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & -3 \\ 0 & 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

(a) Use Gram-Schmidt procedure to find the rectangular QR factorization of A.

(b) Use the QR factors from part (a) to determine the least squares solution to $Ax = b$.

3. Let $\mathcal{X}$ and $\mathcal{Y}$ be subspaces of $\mathbb{R}^3$ whose respective bases are

$$\mathcal{B}_{\mathcal{X}} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right\} \text{ and } \mathcal{B}_{\mathcal{Y}} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$$

(a) Explain why $\mathcal{X}$ and $\mathcal{Y}$ are complementary subspaces of $\mathbb{R}^3$.

(b) Determine the projector P onto $\mathcal{X}$ along $\mathcal{Y}$ as well as the complementary projector Q onto $\mathcal{Y}$ along $\mathcal{X}$.

(c) Determine the projection of $v = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ onto $\mathcal{Y}$ along $\mathcal{X}$.

4. If $A_{n\times n}$ is nonsingular, and if $\mathbf{c}$ and $\mathbf{d}$ are $n \times 1$ columns. Prove the following

(a) $\det(I + \mathbf{c}\mathbf{d}^T) = 1 + \mathbf{d}^T\mathbf{c}$

(b) $\det(A + \mathbf{c}\mathbf{d}^T) = \det(A)(1 + \mathbf{d}^T A^{-1}\mathbf{c})$.

(c) Using (a) and (b), find $\det(A)$, where $A = \begin{pmatrix} 1+\lambda_1 & 1 & \cdots & 1 \\ 1 & 1+\lambda_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+\lambda_n \end{pmatrix}$

and $\lambda_i \neq 0$.

5. Use Gerschgorin's theorem that show that $A = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 12 & 0 & -4 \\ 1 & 0 & -1 & 0 \\ 0 & 5 & 0 & 0 \end{pmatrix}$ must have at least two real eigenvalues.

6. If A is nilpotent ($A^k = 0$ for some k), show that $\text{Trace}(A) = 0$

   Hint: Consider $\sigma(A)$

7. Find $\kappa_1(A), \kappa_F(A)$ and $\kappa_\infty(A)$ of $A = \begin{pmatrix} 10^{-4} & 1 \\ -1 & 2 \end{pmatrix}$ and tell whether it is ill-condition or well-condition matrix.

8. Consider $A = \begin{pmatrix} \delta & 1 \\ 1 & 1 \end{pmatrix}$, where $0 < \delta \leq 1$

(a) Find the LU factorization of A without pivoting and discuss whether A is well-conditioned or ill-conditioned matrix.

(b) Repeat (a) with pivoting.