King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 460
Major Exam I, Semester II, 2015-2016
Net Time Allowed: 120 minutes

Name:————————————————————————————
ID:————————————Section:——————Serial:————————

Q#	Marks	Maximum Marks
1		12
2		10
3		6
4		11
5		10
6		5
7		6
Total		60

1. Write clearly.
2. Show all your steps.
3. No credit will be given to wrong steps.
4. Do not do messy work.
5. Calculators and mobile phones are NOT allowed in this exam.
6. Turn off your mobile.

1. Let $A = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 0 & -2 \\ 2 & 2 & 4 \end{bmatrix}$

   (a) Find elementary matrices E_1, E_2, E_3 and P such that

   $$E_3 E_2 E_1 P A = U.$$

   (b) Using (a) compute the LU factorization of the matrix PA (i.e show that $A = PLU$)

   (c) Using (b), solve the system $Ax = b$, where $b = (-3, 1, 0)$.

2. Find bases and their dimensions for the four fundamental subspaces of

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 2 \end{bmatrix}$$

3. Let $\mathcal{X}$ and $\mathcal{Y}$ are two subspaces of a vector space $\mathcal{V}$, show the following

   (a) $\mathcal{X} + \mathcal{Y} = \{x + y \mid x \in \mathcal{X}, y \in \mathcal{Y}\}$ is a subspace of $\mathcal{V}$.

   (b) $\dim\ (\mathcal{X} + \mathcal{Y}) = \dim(\mathcal{X}) + \dim(\mathcal{Y}) - \dim(\mathcal{X} \bigcap \mathcal{Y})$

   (c) For any conformal matrices A and B (use part(b)) to show

$$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$$

4. Identify all subspaces of $\mathbb{R}^2$ that are invariant under $A = \begin{bmatrix} -9 & 4 \\ -24 & 11 \end{bmatrix}$

5. If $\mathbf{P}$ is the projector that maps each $v = (x, y, z) \in \mathbb{R}^3$ to its orthogonal projection $\mathbf{P}(v) = (x, y, 0)$ in the xy-plane.

   (a) Calculate $[\mathbf{P}]_{\mathcal{B}\mathcal{B}'}$ where the bases $\mathcal{B}$ and $\mathcal{B}'$ are

$$\mathcal{B} = \left\{ u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

   and

$$\mathcal{B}' = \left\{ v_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

   (b) Find $[\mathbf{P}(w)]_{\mathcal{B}'}$, where $w = (-1, 2, -1)$.

6. Find A^{3000}, given

$$A = \begin{bmatrix} 1 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

Hint: A square matrix C is said to be idempotent when it has the property that $C^2 = C$. Make use of idempotent submatrices in A.

7. Let S be a skew-symmetric matrix with real entries.

   (a) Prove that $(I - S)$ is nonsingular. **Hint:** $x^T x = 0 \to x = 0$.

   (b) If $A = (I + S)(I - S)^{-1}$, show that $A^{-1} = A^T$.