KFUPM - Department of Mathematics and Statistics
MATH 345, Term 152
Final Exam (Out of 140), Duration: 180 minutes
NAME:
ID:

Solve the following Exercises.

Exercise 1 (15 points 5-5-5): Let G be a finite cyclic group.
(1) If 5 divides $|G|$, How many elements of order 5 does G have? Justify.
(2) If 9 divides $|G|$, How many elements of order 9 does G have? Justify.
(3) If a is an element of G with $|a| = 9$. Find all other elements of order 9 of G. Justify.

Exercise 2 (15 points 5-5-5): Let $G = \mathbb{Z}_{16}$ and $G' = \mathbb{Z}_2 \bigoplus \mathbb{Z}_8$.
(1) Find all elements of order 4 of G.
(2) Find all elements of order 4 of G'.
(3) Are G and G' isomorphic? Justify.

Exercise 3 (25 points 5-5-5-5-5): Let R be an integral domain and M and N two distinct maximal ideals of R.
(1) Prove that $M^2 + N^2 = R$.
(2) Prove that $MN = M \cap N$.
(3) Let $\phi : R \longrightarrow R/M \times R/N$ be the ring homomorphism defined by $\phi(a) = (\bar{a}[M], \bar{a}[N])$ (where $\bar{a}[M]$ is the class of a mod M, $\bar{a}[N]$ is the class of a mod N). Prove that $ker(\phi) = MN$.
(4) Prove that ϕ is onto. [Hint, if $(\bar{x}[M], \bar{y}[N])$ is an element of $R/M \times R/N$, use $1 = a + b$ for some $a \in M$ and $b \in N$ to find an element $z \in R$ such that $\phi(z) = (\bar{x}[M], \bar{y}[N])$.
(5) Prove that R/MN is isomorphic to $R/M \times R/N$. (Hint: Use the First Isomorphism Theorem applied to ϕ)

Exercise 4 (10 points 5-5): (1) Prove that the additive groups $(\mathbb{Z}[\sqrt{2}], +)$ and $(\mathbb{Z}[\sqrt{3}], +)$ are isomorphic.
(2) Prove that the rings $(\mathbb{Z}[\sqrt{2}], +, \times)$ and $(\mathbb{Z}[\sqrt{3}], +, \times)$ are not isomorphic.

Exercise 5 (20 points, 5-5-5-5): Let D be an integral domain and K a field.
(1) Prove that $K[X]$ is a PID (Principal Ideal Domain).
(2) Prove that $D[X]$ is a PID if and only if D is a field.
(3) Let $I = \{f(X) \in K[X] | f(a) = 0 \text{ for all } a \in K\}$. Prove that I is an ideal of $K[X]$.
(4) Assume that K is finite. Find a monic polynomial $g \in K[X]$ such that $I = (g)$.

Exercise 6 (25 points, 5-5-5-5-5): (1) Prove that $f = X^3 + X^2 + 1$ is irreducible over $\mathbb{Z}$.
(2) Prove that $X^7 + 343X^5 + 7X^3 + 49X + 14$ is irreducible over Q.
(3) Prove that $Q[X]/(X^2 - 2)$ is isomorphic to $Q[\sqrt{2}]$. ($Q[\sqrt{2}] = \{a + b\sqrt{2} | a, b \in \mathbb{Q}$)
(4) Prove that $Q[\sqrt{2}]$ is a field.
(5) Prove that the ideal $(X^2 - 2)$ is a maximal ideal of $\mathbb{Q}[X]$.

Exercise 7 (20 points, 5-5-5-5): Let $D = \mathbb{Z}[\sqrt{-5}]$.
(1) Prove that $1 + \sqrt{-5}$ and $1 - \sqrt{-5}$ are irreducible but not prime.
(2) Prove that 2 and 3 are irreducible in D.
(3) Find two factorizations of 6 in D.
(4) Is D a UFD (Unique Factorization Domain)? Justify.

Exercise 8 (10 points, 5-5): Let R be a PID, $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \ldots$ be a chain of ideals of R and $I = \bigcup_{n \geq 1} I_n$.
(1) Prove that I is an ideal of R.
(2) Prove that there is there is a positive integer n_0 such that $I_n = I_{n_0}$ for all $n \geq n_0$.