KFUPM - Department of Mathematics and Statistics
MATH 345, Term 132
Exam II (Out of 100), Duration: 120 minutes
NAME:
ID:

Solve the following Exercises.

Exercise 1 (20 points 8-4-8):
Let G be a finite abelian group of order 909.
(1) Prove that G is the internal direct product of two groups H and K to be determined.
(2) Find the orders of H and K.
(3) Describe completely the groups H and K. (Hint: Every abelian group of finite order is isomorphic to a group of the form $\mathbb{Z}_{p_1^{n_1}} \bigoplus \cdots \bigoplus \mathbb{Z}_{p_r^r})$

Exercise 2 (20 points 6-8-8):
Let $\mathbb{Z}[\sqrt{3}]$ be the additive group of all elements of the form $\{x+y\sqrt{3}|x, y \text{ integers }\}$. For a fixed integer a, we define the following two maps ϕ_a and ψ_a :$\mathbb{Z}[\sqrt{3}] \longrightarrow \mathbb{Z}[\sqrt{3}]$ by $\phi_a(x+y\sqrt{3}) = x + ay + y\sqrt{3}$ and $\psi_a(x+y\sqrt{3}) = x + ay - y\sqrt{3}$.
(1) Prove that ϕ_a and ψ_a are automorphisms of the additive group $(\mathbb{Z}[\sqrt{3}], +)$ with $\phi(1) = \psi(1) = 1$.
(2) Use (1) to find all automorphisms ϕ of $\mathbb{Z}[\sqrt{3}]$ satisfying $\phi(1) = 1$.
(3) Find two infinite groups $H \subsetneqq G$ such that $Aut(H)$ is finite and $Aut(G)$ is infinite.

Exercise 3 (20 points 6-4-4-6): Let G and G' be groups, H a normal subgroup of G and $\phi : G \longrightarrow G'$ a group homomorphism.
(1) Prove that if ϕ is onto, then $\phi(H)$ is a normal subgroup of G'.
Let $G = \mathbb{R}^*$ be the multiplicative group of nonzero real numbers, $H = \mathbb{Q}^*$ its normal subgroup of nonzero rational numbers, $G' = SL_2(\mathbb{R}) = \{$ all 2×2 matrices A with $detA \neq 1\}$ and $\phi : G \longrightarrow G'$ defined by $\phi(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$.
(2) Prove that ϕ is a group homomorphism.
(3) Find $Ker\phi$ and $Im(\phi)$. Is ϕ a one-to-one? Is ϕ onto?
(4) Prove that $\phi(\mathbb{Q}^*)$ is not normal in G'.

Exercise 4 (20 points 5-5-5-5) Let G, G' and H be three groups (denoted multiplicatively).
(1) Prove that $G \bigoplus G'$ is isomorphic to $G' \bigoplus G$.
(2) Prove that if G is isomorphic to H, then $G \bigoplus G'$ is isomorphic to $H \bigoplus G'$.
(3) Assume that $H \subseteq G \subseteq G'$. Prove that $(G'/H)/(G/H)$ is isomorphic to G'/G.
(4) Find all isomorphic groups among the following groups: $\mathbb{Z}_{84}$, $\mathbb{Z}_2 \bigoplus \mathbb{Z}_2 \bigoplus \mathbb{Z}_3 \bigoplus \mathbb{Z}_7$, $\mathbb{Z}_3 \bigoplus \mathbb{Z}_4 \bigoplus \mathbb{Z}_7$, $\mathbb{Z}_{12} \bigoplus \mathbb{Z}_7$, $\mathbb{Z}_{14} \bigoplus \mathbb{Z}_6$.

Exercise 5 (20 points, 5-5-5-5):
(1) Find all subgroups of index 2 of the multiplicative group $\mathbb{R}^*$.
(2) Prove that the additive group $\mathbb{Q}$ has no subgroup of finite index.
(3) Prove that the additive group $\mathbb{Q}$ has no maximal subgroup (i.e. a proper subgroup H such that there is no other subgroup contained properly between H and $\mathbb{Q}$).[Hint: Every abelian group G whose only subgroups are itself and the identity is isomorphic to $\mathbb{Z}_p$ for some prime p.
(4) If $(\mathbb{Q}, +)$ a maximal subgroup of the additive group $(\mathbb{R}, +)$?