KFUPM - Department of Mathematics and Statistics
MATH 345, Term 152
Exam I (Out of 100), Duration: 120 minutes

NAME:
ID:

Solve the following Exercises.

Exercise 1 (15 points: 5-5-5). Let G be the multiplicative group. Prove that G is abelian under each one of the following conditions:
(1) $(ab)^2 = a^2b^2$ for every $a, b \in G$.
(2) For every a, b, c in G, $ab = ca$ implies that $b = c$.
(3) The map $f : G \longrightarrow G$ defined by $f(x) = x^{-1}$ is an isomorphism.

Exercise 2 (20 points 7-6-7). Let G be a finite group with order p^n where p is a positive prime integer and n is a positive integer.
(1) Prove that the center $\mathcal{Z}(G)$ of G cannot be of order p^{n-1}.
(2) Assume that G is cyclic and has exactly three subgroups, $\{e\}$, H and G. Find n.
(3) Find a noncyclic group G such that $|G| = p^2$ for some positive integer prime p and G has exactly 4 proper subgroups.

Exercise 3 (20 points 5-5-5-5). Let G be a cyclic group with order $n = 16$.
(1) Prove that G cannot have more that 8 elements x such that $x^8 = e$.
(2) Prove that G cannot have more that 4 elements y such that $y^4 = e$.
(3) Find a noncyclic group H of order 4 which has 3 elements of order 2 but no element of order 4.
(1) Find an infinite abelian multiplicative group which has exactly two elements of order 4.

Exercise 4 (15 points 7-7-6). Let G be an abelian finite group such that $|G| = 10$.
(1) Is G a cyclic group? Justify.
(2) If G is an abelian group with $G = pq$ where $p \neq q$ are prime integers, is G a cyclic group?
(3) If $|G| = p^2$ for some positive prime integer p, does G contain an element of order p^2? Is G cyclic? Justify.

Exercise 5 (15 points 5-5-5). (1) Let G be a group of permutations of a set A and let $a \in A$. Set $stab(a) = \{\sigma \in G | \sigma(a) = a\}$. Prove that $stab(a)$ is a subgroup of G. In the symmetric group S_4, find:
(2) A cyclic group of order 4.
(3) A non-cyclic group of order 4.

Exercise 6 (15 points 5-5-5).
(1) Prove that the additive group $(\mathbb{Q}, +)$ cannot be isomorphic to the multiplicative group $(\mathbb{R}^{+*}, .)$
(2) Show that $\mathbb{Z}$ has infinitely many subgroups isomorphic to $\mathbb{Z}$.
(3) Find two groups G and H such that G is not isomorphic to H but $Aut(G)$ is isomorphic to $Aut(H)$.