(9) Solve the Initial Value Problem: [11 points]

$y''' + 2y'' + 9y' + 18y = 0$ subject to $y(0) = 0,\quad y'(0) = 2 \quad and \quad y''(0) = -13$

Given that one solution of the differential equation is $y(x) = e^{-2x}$.

1pt

The characteristic eqn of the differential eqn is: $r^3+2r^2+9r+18=$

Since we're given the information that $y(x)=e^{-2x}$ is a solution

we know that $r=-2$ is a root of the ch. eqn. 1pt

By dividing the ch. eqn by $r+2$, we can find the other roots

$$\begin{array}{r|l} & r^2+9 \\ \hline r+2 & r^3+2r^2+9r+18 \\ & -\left(r^3+2r^2\right) \\ \hline & 9r+18 \\ & -\left(9r+18\right) \\ \hline & 0 \end{array}$$

$r^3+2r^2+9r+8 = (r+2)(r^2+9)=0$ 1pt

So the roots of the ch. eqn are $r_1=-2$ $r_{2,3}=\pm 3i$ 1pt

Then the general solution is of the form:

$y(x)=C_1e^{-2x}+C_2\cos(3x)+C_3\sin(3x)$. 3pts

To find C_1, C_2, and C_3, we use the initial condicions:

$y'(x)= -2c_1e^{-2x} -3c_2\sin(3x) + 3c_3\cos(3x)$ 1pt

$y''(x)= 4c_1e^{-2x} -9c_2\cos(3x) -27c_3\sin(3x)$. 1pt

$$\left.\begin{aligned} y(0) &= c_1+c_2=0 \\ y'(0) &= -2c_1+3c_3=2 \\ y''(0) &= 4c_1-9c_2=-13 \end{aligned}\right| \Rightarrow \quad \left.\begin{aligned} c_1 &= -1 \\ c_2 &= 1 \\ c_3 &= 0 \end{aligned}\right|$$ 1pt

Then the general solution is:

$y(x)= -e^{-2x} + \cos(3x)$ 1pt

(10) If the complimentary solution of the differential equation [10 points]

$y'' - 2y' + 17y = xe^x \cos 4x$ is $y_c = e^x (A\cos 4x + B\sin 4x)$,

write the form of the particular solution of the differential equation. (Do not determine constants).

Call $f(x) = xe^x \cos 4x$.

Based on $f(x)$, the first version of a particular solution should be $y_p(x) = \underline{xe^x(A\cos 4x + B\sin 4x)} + e^x(C\cos 4x + D\sin 4x)$

3pts 3 pts

We observe that $e^x(C\cos 4x + D\sin 4x)$ is duplicating in y_p and y_c. 1 pt

To remove the duplication, we multiply it by x. 2pts

Then the particular solution is of the form:

$y_p(x) = x^2e^x(A\cos 4x + B\sin 4x) + xe^x(C\cos 4x + D\sin 4x)$ 1pt

(11) Using the method of variation of parameters solve the Initial Value Problem [13 points]

$$\left(x^2-1\right)y''+4xy'+2y=\frac{2}{x+1},\quad y(0)=-1,\quad y'(0)=-5$$

Given that $y_1=\frac{1}{x-1}$ *and* $y_2=\frac{1}{x+1}$ are solutions of the homogenous part.

We rewrite the differential eqn: $y''+\frac{4x}{x^2-1}y'+\frac{2}{x^2-1}y=\frac{2}{(x^2-1)(x+1)}$

$$W(y_1,y_2)=\begin{vmatrix}(x-1)^{-1} & (x+1)^{-1}\\ -1(x-1)^{-2} & -1(x+1)^{-2}\end{vmatrix}=(x-1)^{-2}(x+1)^{-1}-(x-1)^{-1}(x+1)^{-2}\quad \text{2pts}$$

$$=(x-1)^{-2}(x+1)^{-2}\left(x+1-(x-1)\right)$$

$$=2(x-1)^{-2}(x+1)^{-2}\quad \text{1pt}$$

$$y_p=\left(-\int\frac{(x+1)^{-1}\cdot 2(x+1)^{-1}(x-1)^{-1}}{2(x-1)^{-2}(x+1)^{-2}}dx\right)\frac{1}{x-1}+\left(\int\frac{(x-1)^{-1}2(x+1)^{-2}(x-1)^{-1}}{2(x-1)^{-2}(x+1)^{-2}}dx\right)\frac{1}{x+1}\quad \text{1pt}\quad \text{1pt}$$

$$y_p=\left(-\int\frac{x-1}{x+1}dx\right)\frac{1}{x-1}+\left(\int dx\right)\frac{1}{x+1}\quad \text{1pt}\quad \text{1pt}$$

$$y_p=\left(-x+2\ln(x+1)\right)\frac{1}{x-1}+\frac{x}{x+1}=\frac{2\ln(x+1)}{x-1}-\frac{2x}{x^2-1}\quad \text{1pt}$$

Then a general solution is of the form:

$$y(x)=\frac{c_1}{x-1}+\frac{c_2}{x+1}+\frac{2\ln(x+1)}{x-1}-\frac{2x}{x^2-1}\quad \text{1pt}$$

To find c_1 and c_2, we use the initial conditions

$$y'(x)=\frac{-c_1}{(x-1)^2}-\frac{c_2}{(x+1)^2}+\frac{\frac{2}{x+1}\cdot(x-1)+2\ln(x+1)}{(x-1)^2}-\frac{2(x^2-1)-4x^2}{(x^2-1)^2}\quad \text{1pt}$$

$$y(0)=-c_1+c_2=-1.$$

$$y'(0)=-c_1-c_2-2+2=-5$$

$$\Rightarrow\quad \begin{matrix}-c_1+c_2=-1\\ -c_1-c_2=-5\end{matrix}\quad\Rightarrow\quad \begin{matrix}c_1=3 \ \text{1pt}\\ c_2=2 \ \text{1pt}\end{matrix}$$

Then the general solution is

$$y(x)=\frac{3}{x-1}+\frac{2}{x+1}+\frac{2\ln(x+1)}{x-1}-\frac{2x}{x^2-1}.\quad \text{1pt}$$

(12) Suppose that the matrix $A=\begin{pmatrix}2&2&3\\1&2&1\\2&-2&1\end{pmatrix}$ is diagonalizable with eigenvalues

$\lambda_1=-1,\quad \lambda_2=2,\quad \lambda_3=4.$ Find two matrices P & D so that $A=PDP^{-1}$ [12 points]

The eigenvalues are already given. We need to find next eigenvectors.

eigenvector assoc. to $\lambda=-1$:

We solve the system $(A+I)v=0$.

$$\begin{bmatrix}3&2&3\\1&3&1\\2&-2&2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\xrightarrow[\text{row operations}]{\text{after}}\begin{bmatrix}1&3&1\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\Rightarrow \begin{matrix}a+3b+c=0\\b=0.\end{matrix}$$

3pts

Choose $a=t$. Then $c=-t$.

The eigenvector assoc. to $\lambda=-1$ is $V_1=\begin{bmatrix}1\\0\\-1\end{bmatrix}$.

eigenvector assoc. to $\lambda=2$:

We solve the system $(A-2I)v=0$.

$$\begin{bmatrix}0&2&3\\1&0&1\\2&-2&-1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\xrightarrow[\text{row operations}]{\text{after}}\begin{bmatrix}1&0&1\\0&2&3\\0&0&0\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\Rightarrow \begin{matrix}a+c=0\\2b+3c=0\end{matrix}$$

3pts

Choose $c=2t$. Then $b=-3t$ and $a=-2t$.

The eigenvector assoc to $\lambda=2$ is $V_2=\begin{bmatrix}-2\\-3\\2\end{bmatrix}$

eigenvector assoc. to $\lambda=4$:

We solve the system $(A-4I)=0$.

$$\begin{bmatrix}-2&2&3\\1&-2&1\\2&-2&-3\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\xrightarrow[\text{row operations}]{\text{after}}\begin{bmatrix}1&-2&1\\0&-2&5\\0&0&0\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\Rightarrow \begin{matrix}a-2b+c=0\\-2b+5c=0\end{matrix}$$

3pts

Choose $c=2t$. Then $b=5t$ and $a=8t$

The eigenvector assoc to $\lambda=4$ is $V_3=\begin{bmatrix}8\\5\\2\end{bmatrix}$.

Then $P=\begin{bmatrix}1&-2&8\\0&-3&5\\-1&2&2\end{bmatrix}$ 2pts and $D=\begin{bmatrix}-1&0&0\\0&2&0\\0&0&4\end{bmatrix}$ 1pt

(13) Consider the matrix $A=\begin{pmatrix}1 & 2 & 2\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}$, use Cayley-Hamilton Theorem to find

A^3 *and* A^{-1}. [10 points]

We find the characteristic polynomial of A

$$p(\lambda)=|A-\lambda I|=\begin{vmatrix}1-\lambda & 2 & 2\\ 0 & 1-\lambda & 2\\ 0 & 0 & 1-\lambda\end{vmatrix} \text{ 1pt } =(1-\lambda)^3 \text{ 1pt}$$

Then by Cayley-Hamilton Theorem, $p(A)=(I-A)^3=0$ 3pts

$$I-3A+3A^2-A^3=0 \Rightarrow A^3=3A^2-3A+I \text{ 1pt}$$

$$A^2=\begin{bmatrix}1 & 2 & 2\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 4 & 8\\ 0 & 1 & 4\\ 0 & 0 & 1\end{bmatrix} \text{ 1pt}$$

$$A^3=3\begin{bmatrix}1 & 4 & 8\\ 0 & 1 & 4\\ 0 & 0 & 1\end{bmatrix}-3\begin{bmatrix}1 & 2 & 2\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}+\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 6 & 18\\ 0 & 1 & 6\\ 0 & 0 & 1\end{bmatrix} \text{ 1pt}$$

To find A^{-1}, we solve $p(A)=0$ for I.

$$I=3A-3A^2+A^3 \text{ 1pt}$$

We multiply both sides by A^{-1}.

$$A^{-1}=3I-3A+A^2$$

$$=3\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}-3\begin{bmatrix}1 & 2 & 2\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}+\begin{bmatrix}1 & 4 & 8\\ 0 & 1 & 4\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & -2 & 2\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix} \text{ 1pt}$$

(14) Consider the homogenous system $X'(t)=\begin{pmatrix}6 & 4 & 4\\ -7 & -2 & -1\\ 7 & 4 & 3\end{pmatrix}X(t)$. [10 points]

a) Verify that $X_1(t)=\begin{bmatrix}1\\-1\\1\end{bmatrix}e^{6t}$ is a solution of the system.

b) Verify that the solutions $X_1(t)=\begin{bmatrix}1\\-1\\1\end{bmatrix}e^{6t}$, $X_2(t)=\begin{bmatrix}1\\-2\\1\end{bmatrix}e^{2t}$, $X_3(t)=\begin{bmatrix}0\\-1\\1\end{bmatrix}e^{-t}$

are linearly independent.

c) Find a general solution of the system.

a) $X_1'(t)=\begin{bmatrix}6\\-6\\6\end{bmatrix}e^{6t}$ 1pt

$$\begin{bmatrix}6 & 4 & 4\\ -7 & -2 & -1\\ 7 & 4 & 3\end{bmatrix}\begin{bmatrix}1\\-1\\1\end{bmatrix}e^{6t} = \begin{bmatrix}6-4+4\\ -7+2-1\\ 7-4+3\end{bmatrix}e^{6t} = \begin{bmatrix}6\\-6\\6\end{bmatrix}e^{6t}$$ 1pt

So $X_1(t)$ is a solution.

b)

$$W(X_1,X_2,X_3)=\begin{vmatrix}e^{6t} & e^{2t} & 0\\ -e^{6t} & -2e^{2t} & -e^{-t}\\ e^{6t} & e^{2t} & e^{-t}\end{vmatrix} = e^{6t}\begin{vmatrix}-2e^{2t} & -e^{-t}\\ e^{2t} & e^{-t}\end{vmatrix} - e^{2t}\begin{vmatrix}-e^{6t} & -e^{-t}\\ e^{6t} & e^{-t}\end{vmatrix}$$ 2pts

$$= e^{6t}(-2e^{t}+e^{t}) - e^{2t}(-e^{5t}+e^{5t}) = -e^{7t} \neq 0$$ 1pt 1pt

Then X_1, X_2, and X_3 are linearly independence. 1pt

c) $X(t)=c_1X_1(t)+c_2X_2(t)+c_3X_3(t)$

$x_1(t)=c_1e^{6t}+c_2e^{2t}$ 1pt

$x_2(t)=-c_1e^{6t}-2c_2e^{2t}-e^{-t}$ 1pt

$x_3(t)=e^{6t}+e^{2t}+e^{-t}$ 1pt

(15) Find a general solution of the system $X' = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} X$. [13 points]

We first find all eigenvalues and eigenvectors.

Call $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

$|A-\lambda I| = \begin{vmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix}$ (1pt) $= -\lambda^3 + 3\lambda^2 - 4\lambda + 2 = 0$ (1pt)

eigenvalues are $\lambda_1 = 1$ (1pt) $\lambda_{2,3} = 1 \pm i$ (1pt)

eigenvector assoc to $\lambda_1 = 1$:

We solve the system $(A-I)v=0$ and find that $V_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ (2pts)

eigenvector assoc to $\lambda_2 = 1+i$:

We solve the system $(A-(1+i)I)v=0$ and find that $V_2 = \begin{bmatrix} -i \\ 1 \\ 1 \end{bmatrix}$ (2pts)

eigenvector assoc to $\lambda_3 = 1-i$: is complex conjugate of the eigenvector assoc to $\lambda_2 = 1+i$. So $V_3 = \begin{bmatrix} i \\ 1 \\ 1 \end{bmatrix}$. (2pts)

a general solution is:

$$X(t) = c_1 e^t V_1 + c_2 e^{(1+i)t} V_2 + c_3 e^{(1-i)t} V_3$$

$$x_1(t) = (-c_2+c_3)i\, e^t \cos t + (c_2+c_3) e^t \sin t$$

$$x_2(t) = 2c_1 e^t + (c_2+c_3) e^t \cos t + (-c_2+c_3) i e^t \sin t.$$

$$x_3(t) = c_1 e^t + (c_2+c_3) e^t \cos t + (-c_2+c_3) i e^t \sin t.$$

(3pts)

(16) Find a general solution of the system $X' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} X$. [13 points]

We first find all eigenvalues and eigenvectors.

Call $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = \lambda^2(1-\lambda) = 0$ (1pt)

eigenvalues are $\lambda_1 = 0$ (multiplicity 2) (1pt) and $\lambda_2 = 1$. (1pt)

eigenvector assoc. $\lambda_1 = 0$

We solve the system $Av = 0$. and find that $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (1pt)

Then the eigenvalue $\lambda_1 = 0$ is defective (1pt) with defect $= 2 - 1 = 1$.

We need to find a length 2 chain of generalized eigenvectors generated by v_1. That is a set $\{u_1, u_2\}$ of vectors where $A^2 u_2 = 0$ and $Au_2 = u_1$

$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (1pt) so choose $u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. (1pts)

$u_1 = Au_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (1pts)

eigenvector assoc. to $\lambda_2 = 1$:

We solve the system $(A - I)v = 0$ and find that $V_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(1pt)

Then the general solution is:

$$X(t) = c_1 u_1 + c_2(u_1 t + u_2) + c_3 e^t v_2 . \rightsquigarrow \text{(2pts)}$$

$$\left.\begin{aligned} x_1(t) &= c_1 + c_2(t+1) \\ x_2(t) &= c_2 \\ x_3(t) &= c_3 e^t \end{aligned}\right| \longrightarrow \text{(2pts)}$$