King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
**(Math 260)**

**Second Major Exam**
**Term 152**
**Thursday, March 31, 2016**
Net Time Allowed: 100 minutes

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|---|---|
| Name: | |
| ID: | |
| Section No: | |
| Instructor's Name | |

**(Show all your steps and work)**

| Question # | Marks |
|---|---|
| 1 | 10 |
| 2 | 10 |
| 3 | 12 |
| 4 | 12 |
| 5 | 14 |
| 6 | 12 |
| 7 | 8 |
| 8 | 8 |
| 9 | 14 |
| **Total** | **/100** |

(1) If $A = \begin{bmatrix} 3 & x \\ -5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 7 \\ 5 & y \end{bmatrix}$, then find $x$ and $y$ so that $AB = BA$.

[10 points]

$$AB = \begin{bmatrix} 3 & x \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 7 & 7 \\ 5 & y \end{bmatrix} = \begin{bmatrix} 21+5x & 21+xy \\ -30 & -35+y \end{bmatrix} \quad \underline{\underline{2\text{ pts}}}$$

$$BA = \begin{bmatrix} 7 & 7 \\ 5 & y \end{bmatrix} \begin{bmatrix} 3 & x \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} -14 & 7x+7 \\ 15-5y & 5x+y \end{bmatrix} \quad \underline{\underline{2\text{ pts}}}$$

Since $AB = BA$, we have the following equations:

$21+5x = -14$ $\quad \underline{\underline{2\text{ pts}}}$

$21+xy = 7x+7$

$15-5y = -30$ $\quad \underline{\underline{2\text{ pts}}}$

$-35+y = 5x+y$

From the 1st equation, we find $x = -7$ $\quad \underline{\underline{1\text{ pt}}}$

From the 3rd equation, we find $y = 9$ $\quad \underline{\underline{1\text{ pt}}}$

One can verify that these values ~~ef~~ $x$ and $y$ satisfy the 2nd and the 4th equations, as well.

(2) Let $A = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 & 2 \\ 1 & -4 & -3 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & -2 \\ 3 & -1 \end{bmatrix}$.

Find the matrix $X$ so that $CX = AB^T$. [10 points]

$$B^T = \begin{bmatrix} 4 & 1 \\ 0 & -4 \\ 2 & -3 \end{bmatrix} \qquad \underline{\underline{2\text{ pts}}}$$

$$AB^T = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 0 & -4 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 12 & 0 \\ 8 & -24 \end{bmatrix} \qquad \underline{\underline{2\text{ pts}}}$$

$$X = C^{-1}(AB^T) \qquad \underline{\underline{2\text{ pts}}}$$

$$C^{-1} = \frac{1}{2.(-1)-3.(-2)} \begin{bmatrix} -1 & 2 \\ -3 & 2 \end{bmatrix} = \underbrace{\frac{1}{4}}_{\underline{\underline{1\text{ pt}}}} \underbrace{\begin{bmatrix} -1 & 2 \\ -3 & 2 \end{bmatrix}}_{\underline{\underline{1\text{ pt}}}}$$

Then $X = \frac{1}{4} \begin{bmatrix} -1 & 2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 12 & 0 \\ 8 & -24 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & -6 \end{bmatrix}$

$$X = \begin{bmatrix} 1 & -12 \\ -5 & -12 \end{bmatrix} \qquad \underline{\underline{2\text{ pts}}}$$

(3) Use Cramer's rule to solve the system [12 points]

$$x + 2y + z = 0,$$
$$3y + 2z = 1,$$
$$x + y + z = 0.$$

The matrix form of the system is

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \underline{\underline{1\ pt}}$$

- $\begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = (3-2) + (4-3) = 2 \quad \underline{\underline{2\ pts}}$

- $\begin{vmatrix} 0 & 2 & 1 \\ 1 & 3 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -1 \quad \underline{\underline{2\ pts}}$

- $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0 \quad \underline{\underline{2\ pts}}$

- $\begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 \quad \underline{\underline{2\ pts}}$

Then $x = \dfrac{\begin{vmatrix} 0 & 2 & 1 \\ 1 & 3 & 2 \\ 0 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 1 \end{vmatrix}} = -\frac{1}{2}$ $\underline{\underline{1\ pt}}$ $\quad y = \dfrac{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 1 \end{vmatrix}} = 0$ $\underline{\underline{1\ pt}}$

$z = \dfrac{\begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & 1 & 1 \end{vmatrix}} = \frac{1}{2} \quad \underline{\underline{1\ pt}}$

(4) Write the vector $w = (7,7,9,11)$, if possible, as a linear combination of the vectors $v_1 = (2,0,3,1)$, $v_2 = (4,1,3,2)$, *and* $v_3 = (1,3,-1,3)$. [12 points]

We need to find $c_1, c_2, c_3$ so that $c_1 v_1 + c_2 v_2 + c_3 v_3 = w$ 2pts

This vector equation can be written as the following:

$$2c_1 + 4c_2 + c_3 = 7$$
$$c_2 + 3c_3 = 7$$
$$3c_1 + 3c_2 - c_3 = 9$$
$$c_1 + 2c_2 + 3c_3 = 11$$

To find $c_1, c_2, c_3$, we solve the above linear system using Gauss-Jordan elimination method.

$$\underbrace{\begin{bmatrix} 2 & 4 & 1 & 7 \\ 0 & 1 & 3 & 7 \\ 3 & 3 & -1 & 9 \\ 1 & 2 & 3 & 11 \end{bmatrix}}_{3\text{ pts}} \xrightarrow{R_1 \leftrightarrow R_4} \begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 1 & 3 & 7 \\ 3 & 3 & -1 & 9 \\ 2 & 4 & 1 & 7 \end{bmatrix} \xrightarrow[R_4 = R_4 - 2R_1]{R_3 = R_3 - 3R_1}$$

3 pts for the work to reduce the sys.

$$\begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 1 & 3 & 7 \\ 0 & -3 & -10 & -24 \\ 0 & 0 & -5 & -15 \end{bmatrix} \xrightarrow{R_3 = R_3 + 3R_2} \begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & -5 & -15 \end{bmatrix} \xrightarrow{R_4 = R_4 - 5R_3} \begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 = -R_3} \begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 = R_1 - 3R_3]{R_2 = R_2 - 3R_3} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 = R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $c_1 = 6$, $c_2 = -2$, $c_3 = 3$ 1 pt each = 3pts

As a linear combination of $v_1, v_2, v_3$

$$w = 6v_1 - 2v_2 + 3v_3.$$ 1 pt

(5) Find a basis and the dimension of the solution space of the homogeneous linear system

$$\begin{aligned} x_1 - 2x_2 - 3x_3 - 16x_4 &= 0, \\ 2x_1 - 4x_2 + x_3 + 17x_4 &= 0, \\ x_1 - 2x_2 + 3x_3 + 26x_4 &= 0. \end{aligned}$$

[~~12~~ 14 points]

First, we solve the system using Gauss-Jordan elimination method.

$$\underbrace{\begin{bmatrix} 1 & -2 & -3 & -16 \\ 2 & -4 & 1 & 17 \\ 1 & -2 & 3 & 26 \end{bmatrix}}_{1\text{ pt}} \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & -2 & -3 & -16 \\ 0 & 0 & 7 & 49 \\ 0 & 0 & 6 & 42 \end{bmatrix} \xrightarrow[R_3 = R_3/6.]{R_2 = R_2/7}$$

3 pts for the work to reduce the sys.

$$\begin{bmatrix} 1 & -2 & -3 & -16 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 1 & 7 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} 1 & -2 & -3 & -16 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 = R_1 + 3R_2}$$

$$\begin{bmatrix} 1 & -2 & 0 & 5 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The reduced linear system is $\begin{cases} x_1 - 2x_2 + 5x_4 = 0 \\ x_3 + 7x_4 = 0 \end{cases}$

We choose $x_4 = s$ and $x_2 = r$. Then $x_3 = -7s$ and $x_1 = 2r - 5s$.

The solution space of the linear system is

$W = \{(x_1, x_2, x_3, x_4) \mid x_1 = 2r - 5s,\ x_2 = r,\ x_3 = -7s,\ x_4 = s\}$. 1 pt for each $= 4$ pts

If $s = 1$ and $r = 0$, $v_1 = (-5, 0, -7, 1)$. 2 pts

If $r = 1$ and $s = 0$, $v_2 = (2, 1, 0, 0)$ 2 pts

A basis of the solution space is $\{v_1, v_2\}$. 1 pt

The dimension of the solution space is 2. 1 pt

(6) Determine whether or not $V$ is a subspace of $R^3$ where: 12 [~~10~~ points]

a) $V$ is the set of all vectors $(x,y,z) \in R^3$ such that $x-y-z=0.1$

Consider the vectors $u=(0.1,0,0)$ and $v=(0,-0.1,0)$ 2 pts

$u$ is in $V$ since $0.1-0-0=0.1$

$v$ is in $V$ since $0-(-0.1)-0=0.1$

However $\underbrace{u+v=(0.1,-0.1,0)}_{1\text{ pt}}$ is not in $V$ since $\underbrace{0.1-(-0.1)+0 \neq 0.1}_{1\text{ pt}}$

Therefore $V$ is not a subspace of $\mathbb{R}^3$ 1 pt

b) $V$ is the set of all vectors $(x,y,z) \in R^3$ such that $y=2x+3z$.

Verification of the closure property:

1 pt [Let $u=(x_1,y_1,z_1)$ and $v=(x_2,y_2,z_2)$ be two vectors in $V$.

Then we have the relations $y_1=2x_1+3z_1$ and $y_2=2x_2+3z_2$

We show that $\underbrace{u+v=(x_1+x_2,y_1+y_2,z_1+z_2)}_{1\text{ pt}}$ is a vector in $V$.

1 pt
$$\begin{aligned} y_1+y_2 &= (2x_1+3z_1)+(2x_2+3z_2) \\ &= (2x_1+2x_2)+(3z_1+3z_2) \\ &= 2(x_1+x_2)+3(z_1+z_2). \end{aligned}$$

Verification of multiplication by scalar

1 pt [Let $u=(x,y,z)$ be a vector in $V$ and $c \in \mathbb{R}$ be a scalar.

Then we have the relation $y=2x+3z$.

Whe show that the vector $\underbrace{cu=(cx,cy,cz)}_{1\text{ pt}}$ is in $V$.

1 pt [$cy=c(2x+3z)=2cx+3cz$.

Then $V$ is a subspace of $\mathbb{R}^3$. 1 pt

(7) Show that the set of vectors $\{(1,1,0),(1,0,1),(0,1,1)\}$ forms a basis for the vector space $R^3$.

[~~10~~ 8 points]

We first verify that the vectors are linearly independent.

3 pts

To that end, it is enough to show that the determinant of $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is not zero.

2 pts

$\det A = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1 - 1 = -2 \neq 0.$

3 pts

Since the set has 3 linearly independent vectors of $\mathbb{R}^3$, it is a basis of $\mathbb{R}^3$.

(8) Find a general solution of the $2^{nd}$ order constant coefficient homogenous linear differential equation

$$y'' + 3y' - 4y = 0 \ .$$

8
[~~10~~ points]

The characteristic eqn of this differential eqn is

$$r^2 + 3r - 4 = 0 . \quad \underline{\underline{2 \text{ pts}}}$$

Its roots are $r = -4$ and $r = 1$. $\underline{\underline{2 \text{ pts}}}$

Since roots are distinct, $\underbrace{\{e^{-4x}, e^{x}\}}_{}$, form a basis of the solution space of $y'' + 3y' - 4y = 0$. $\underline{\underline{2\text{pts}}}$

Then a general solution is of the form

$$y = c_1 e^{-4x} + c_2 e^{x}, \quad c_1, c_2 \in \mathbb{R} . \quad \underline{\underline{2 \text{ pts}}}$$

(9) The complimentary and the particular solutions of the linear 2nd order non homogeneous differential equation $y'' + 4y = 4x$ are given by $y_C = A\cos 2x + B\sin 2x$ *and* $y_P = x$.

a. Verify that the solutions of the homogenous part are linearly independent.

b. Find the solution of the differential equation that satisfies the initial conditions. $y(0) = 3$ *and* $y'(0) = 4$.

[14 points]

a) The solutions of the homogenous part of the differential equation are $\underbrace{y_1 = \cos 2x}_{1\text{pt}}$, and $\underbrace{y_2 = \sin 2x}_{1\text{pt}}$. To verify their independence, we look at their Wronskian.

$$W(y_1, y_2) = \underbrace{\begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}}_{3\text{ pts}} = \underbrace{2\cos^2 2x + 2\sin^2 2x}_{1\text{pt}} = \underbrace{2 \neq 0}_{1\text{pt}}.$$

Therefore $y_1$ and $y_2$ are linearly independent. 1 pt

b) A general solution of the differential equation $y''+4y=4x$ is of the form $y = y_c + y_p = A\cos 2x + B\sin 2x + x$. 2 pts

We find A and B using the initial conditions.

$y = A\cos 2x + B\sin 2x + x \rightsquigarrow y(0) = A = 3$. 1pt

$\underbrace{y' = -2A\sin 2x + 2B\cos 2x + 1}_{1\text{pt}} \rightsquigarrow y'(0) = 2B + 1 = 4 \Rightarrow B = 3/2$. 1 pt

Then the general solution of the D.E satisfying the initial conditions is

$$y(x) = 3\cos 2x + \frac{3}{2}\sin 2x + x. \quad 1\text{ pt}$$