KFUPM, DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 232: FINAL EXAM, SEMESTER (152), MAY 11, 2016

08:00–11:00 am

Name : ....

ID : ....

Exercise	**Points**
1	: **10**
2	: **10**
3	: **10**
4	: **15**
5	: **10**
6	: **10**
7	: **15**
8	: **10**
9	: **15**
10	: **10**
11	: **10**
12	: **15**
Total	: **140**

Exercise 1 (10 pts)**.** Let $f : \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{R}$ be a function defined recursively by:

$$f(0) = 3, \quad f(k+1) = 2f(k) + 2k - 4.$$

Show that

$$f(n) = 2^n - 2n + 2,$$

for each integer ≥ 0.

Exercise 2 (10 pts)**.** Let R be the binary relation defined on $\mathbb{Z}$ by:

$$a \,\mathrm{R}\, b \quad \text{iff} \quad 3 \mid (a^2 - b^2).$$

(1) Show that R is an equivalence relation.
(2) Show that the quotient set $\mathbb{Z}/\mathrm{R}$ is of cardinality 2.

Exercise 3 (10 pts)**.** Prove that the function $f : \mathbb{R} \setminus \{11\} \longrightarrow \mathbb{R} \setminus \{9\}$ defined by $f(x) = \dfrac{9x-1}{x-11}$ is a bijection and find its inverse function f^{-1}.

Exercise 4 (15 pts)**.** Find the number of integers between 1 and 1000 that are neither even nor multiple of 3 nor multiple of 5 nor multiple of 7.

Exercise 5 (10 pts)**.** Show that among 2196 people, at least 6 were born on the same day of the year.

Exercise 6 (10 pts)**.** Find all integers x, y such that

$$142x - 71y = 213.$$

Exercise 7 (15 pts)**.** Let $\mathbb{I} = (0,1)$ be the unit interval. The decimal representation of an $x \in \mathbb{I}$ is

$$x = \sum_{n=1}^{\infty} a_n 10^{-n},$$

where a_n are digits (in $\{0,1,2,3,4,5,6,7,8,9\}$).

(1) We define the function $f : \mathbb{I} \times \mathbb{I} \longrightarrow \mathbb{I}$ by

$$f(\sum_{n=1}^{\infty} a_n 10^{-n}, \sum_{n=1}^{\infty} b_n 10^{-n}) = \sum_{n=1}^{\infty} c_n 10^{-n},$$

where $c_{2n-1} = a_n$ and $c_{2n} = b_n$ for each positive integer n.

Show that f is injective.

(2) Deduce from (1) that $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R}$ have the same cardinality.

Exercise 8 (10 pts)**.** Find all integers x such that

$$15x \equiv 5 \pmod{50}.$$

Exercise 9 (15 pts)**.** Let $*$ be the binary operation defined on $\mathbb{R} \setminus \{1\}$ by :

$$a * b = a + b - ab.$$

Show that $(\mathbb{R} \setminus \{1\}, *)$ is an Abelian group.

Exercise 10 (10 pts)**.** Show that $(0,1) \setminus \{\frac{1}{n} : n \in \mathbb{N}\}$ and $\mathbb{R}$ have the same cardinality.

Exercise 11 (10 pts)**.** Let H be a subgroup of S_4. Show that $\mid H \mid$ belongs to $\{1, 2, 3, 4, 6, 8, 12, 24\}$.

Exercise 12 (15 pts)**.** Let $U(16)$ be the set of all $x \in \mathbb{Z}_{16}$ having an inverse for the multiplication.

(1) List all the elements of $U(16)$.
(2) Give the table of multiplication on $U(16)$.
(3) Find the order of each element of $U(16)$.
(4) Is $U(16)$ a cyclic group? Justify your answer.
(5) Is there a subgroup of $U(16)$ of order 6? Justify your answer.