MATH 202.5 (Term 152)

Quiz 6 (Sects. 8.2 & 8.3) Duration: 30min

Name: **ID number:**

1.)(4pts)Solve the homogeneous linear system $X' = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix} X$.

2.)(3pts)Solve the homogeneous linear system $X' = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} X$.

3.)(3pts)Solve the system $X' = AX + \begin{pmatrix} te^{-t} \\ 1 \end{pmatrix}$, given that $\Phi(t) = \begin{pmatrix} e^t & 2e^{-2t} \\ -3e^t & e^{-2t} \end{pmatrix}$ is a fundamental matrix of $X' = AX$.

1.) $\begin{vmatrix} 5-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - 8\lambda + 16 = 0$

$(\lambda-4)^2 = 0 \Rightarrow \lambda = 4, 4$

$(A-4I)K = 0 \quad \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right) \quad x - y = 0$

$K\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}.$

$X_2 = (tK + P)e^{4t}, \quad (A-4I)P = K$

$\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & -1 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right)$

$x - y = 1 \Rightarrow P\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow X_2 = \left[t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] e^{4t} = \begin{pmatrix} t+1 \\ t \end{pmatrix} e^{4t}$

$$\boxed{X = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2\begin{pmatrix} t+1 \\ t \end{pmatrix} e^{4t}}$$

2.) $\begin{vmatrix} 1-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = 0 \Leftrightarrow (1-\lambda)^2 + 2 = 0$

$\lambda = 1 \pm i\sqrt{2}$

$(A - (1+i\sqrt{2})I)K = 0$

$\left(\begin{array}{cc|c} -i\sqrt{2} & -1 & 0 \\ 2 & -i\sqrt{2} & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -i\sqrt{2} & 0 \\ 0 & -0 & 0 \end{array}\right)$

$2x - i\sqrt{2}y = 0 \Rightarrow K\begin{pmatrix} i\sqrt{2} \\ 1 \end{pmatrix}$

$\Rightarrow K = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}$

$X_1 = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos\sqrt{2}t - \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}\sin\sqrt{2}t\right] e^{t} = \begin{pmatrix} -\sqrt{2}\sin\sqrt{2}t \\ \cos\sqrt{2}t \end{pmatrix} e^{t}$

$X_2 = \left[\begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}\cos\sqrt{2}t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin\sqrt{2}t\right] e^{t} = \begin{pmatrix} \sqrt{2}\cos\sqrt{2}t \\ \sin\sqrt{2}t \end{pmatrix} e^{t}$

$\Rightarrow \boxed{X = c_1 X_1 + c_2 X_2}$

3.) $X = \underbrace{\Phi C}_{X_c} + \underbrace{\Phi \int \Phi^{-1} F \, dt}_{X_p}$

$\Phi^{-1} = \frac{1}{7e^{-t}}\begin{pmatrix} e^{-2t} & -2e^{-2t} \\ 3e^{t} & e^{t} \end{pmatrix} = \frac{1}{7}\begin{pmatrix} e^{-t} & -2e^{-t} \\ 3e^{2t} & e^{2t} \end{pmatrix}$

$\Phi^{-1}F = \frac{1}{7}\begin{pmatrix} te^{-2t} - 2e^{-t} \\ 3te^{t} + e^{2t} \end{pmatrix}$

$\int \Phi^{-1}F = \frac{1}{7}\begin{pmatrix} -\frac{1}{2}(t+\frac{1}{2})e^{-2t} + 2e^{-t} \\ 3(t-1)e^{t} + \frac{1}{2}e^{2t} \end{pmatrix}$

$$\underbrace{\Phi \int \Phi^{-1}F = \frac{1}{7}\begin{pmatrix} (\frac{11}{2}t - \frac{25}{4})e^{-t} + 3 \\ \frac{9}{2}(t-\frac{1}{2})t - \frac{11}{2} \end{pmatrix}}_{X_p}$$

MATH 202.10 (Term 152)

Quiz 6 (Sects. 8.2 & 8.3) Duration: 30min

Name: ID number:

1.)(4pts)Solve the homogeneous linear system $X' = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} X$.

2.)(3pts)Solve the homogeneous linear system $X' = \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix} X$.

3.)(3pts)Solve the system $X' = AX + \begin{pmatrix} te^t \\ 1 \end{pmatrix}$, given that $\Phi(t) = \begin{pmatrix} e^{2t} & e^{-t} \\ -2e^{2t} & 3e^{-t} \end{pmatrix}$ is a fundamental matrix of $X' = AX$.

1.) $\begin{vmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{vmatrix} = 0 \iff \lambda^2 - 6\lambda + 9 = 0$

$(\lambda-3)^2 = 0, \ \lambda = 3, 3$

$(A - 3I)K = 0$

$\left(\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right) \quad x + y = 0 \quad K\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow X_1 = Ke^{3t}$

$X_2 = (tK + P)e^{3t}, \ (A-3I)P = K$

$\left(\begin{array}{cc|c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$

$x + y = -1 \Rightarrow P\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\Rightarrow X_2 = \left[t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right]e^{3t} = \begin{pmatrix} t-1 \\ -t \end{pmatrix} e^{3t}$

$$\boxed{X = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} t-1 \\ -t \end{pmatrix}e^{3t}}$$

2.) $\begin{vmatrix} 2-\lambda & 1 \\ -3 & 2-\lambda \end{vmatrix} = 0 \iff (2-\lambda)^2 + 3 = 0$

$\lambda = 2 \pm i\sqrt{3}$

$(A - (2+i\sqrt{3})I)K = 0$

$\left(\begin{array}{cc|c} -i\sqrt{3} & 1 & 0 \\ -3 & -i\sqrt{3} & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 3 & i\sqrt{3} & 0 \\ 0 & 0 & 0 \end{array}\right)$

$3x + i\sqrt{3}y = 0, \quad K\begin{pmatrix} i\sqrt{3} \\ -3 \end{pmatrix}$

$- \ K = \begin{pmatrix} 0 \\ -3 \end{pmatrix} + i\begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix}$

$X_1 = \left[\begin{pmatrix} 0 \\ -3 \end{pmatrix}\cos\sqrt{3}t - \begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix}\sin\sqrt{3}t\right]e^{2t} = \begin{pmatrix} -\sqrt{3}\sin\sqrt{3}t \\ -3\cos\sqrt{3}t \end{pmatrix}e^{2t}$

$X_2 = \left[\begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix}\cos\sqrt{3}t + \begin{pmatrix} 0 \\ -3 \end{pmatrix}\sin\sqrt{3}t\right]e^{2t} = \begin{pmatrix} \sqrt{3}\cos\sqrt{3}t \\ -3\sin\sqrt{3}t \end{pmatrix}e^{2t}$

$\Rightarrow \boxed{X = c_1X_1 + c_2X_2}$

3.) $X = \underbrace{\Phi C}_{X_c} + \underbrace{\Phi\int\Phi^{-1}F\,dt}_{X_p}$

$\Phi^{-1} = \frac{1}{5e^{t}}\begin{pmatrix} 3e^{-t} & -e^{-t} \\ 2e^{2t} & e^{2t} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3e^{-2t} & -e^{-2t} \\ 2e^{t} & e^{t} \end{pmatrix}$

$\Phi^{-1}F = \frac{1}{5}\begin{pmatrix} 3te^{-t} - e^{-2t} \\ 2te^{2t} + e^{t} \end{pmatrix}$

$\int\Phi^{-1}F = \frac{1}{5}\begin{pmatrix} -3(t+1)e^{-t} + \frac{1}{2}e^{-2t} \\ (t-\frac{1}{2})e^{2t} + e^{t} \end{pmatrix}$

$$\underbrace{\Phi\int\Phi^{-1}F = \frac{1}{5}\begin{pmatrix} -(2t+\frac{5}{2})e^{t} + \frac{3}{2} \\ 9(t+\frac{1}{2})e^{t} + 2 \end{pmatrix}}_{X_p}$$