MATH 202.5 (Term 152)

Quiz 4 (Sects. 4.6 & 4.7) Duration: 20min

**Name:** **ID number:**

**1.)(5pts)Solve the DE:** $y'' + 6y' + 8y = \cos e^{2x}$.

**2.)(5pts)Solve the DE:** $x^3y''' + 3x^2y'' + xy' + y = 0$.

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1) We solve $y'' + 6y' + 8y = 0$

Its auxiliary equation is $m^2 + 6m + 8 = 0$

$m = -4, \quad m = -2$

$\Rightarrow y_c = c_1 e^{-4x} + c_2 e^{-2x}$

We use variation of parameters to find $y_p$.

We have $y_p = u_1 y_1 + u_2 y_2$,

$u_1' = \dfrac{-y_2 f(x)}{W}, \quad u_2' = \dfrac{y_1 f(x)}{W}$,

with $y_1 = e^{-4x}, \quad y_2 = e^{-2x}, \quad f(x) = \cos e^{2x}$

$$W = \begin{vmatrix} e^{-4x} & e^{-2x} \\ -4e^{-4x} & -2e^{-2x} \end{vmatrix} = 2e^{-6x}$$

$$u_1' = \frac{-e^{-2x} \cos e^{2x}}{2e^{-6x}} = -\frac{1}{2} e^{4x} \cos e^{2x}$$

$\Rightarrow u_1 = -\frac{1}{2} \int e^{4x} \cos e^{2x} \, dx$

$w = e^{2x}, \quad dw = 2e^{2x} dx$

$\int e^{4x} \cos e^{2x} dx = \frac{1}{2} \int w \cos w \, dw$

$= \frac{1}{2}(w \sin w + \cos w)$

$\Rightarrow u_1 = -\frac{1}{4}\left(e^{2x} \sin e^{2x} + \cos e^{2x}\right)$

$$u_2' = \frac{e^{-4x} \cos e^{2x}}{2e^{-6x}} = \frac{1}{2} e^{2x} \cos e^{2x}$$

$u_2 = \frac{1}{2} \int e^{2x} \cos e^{2x} dx$

$= \frac{1}{4} \sin e^{2x}$.

$\Rightarrow y_p = -\frac{1}{4} e^{-4x} \cos e^{2x}$.

and $\underline{y = y_c + y_p}$

2.) $y = x^m$

The auxiliary equation of the DE is $m(m-1)(m-2) + 3m(m-1) + m + 1 = 0$

$m^3 + 1 = 0$

$(m+1)(m^2 - m + 1) = 0$

$m = -1, \quad \Delta = 1 - 4 = -3$

$m = \dfrac{1 \pm i\sqrt{3}}{2}$

$$\Rightarrow \boxed{y = c_1 x^{-1} + x^{\frac{1}{2}}\left(c_2 \cos\left(\frac{\sqrt{3}}{2} \ln x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2} \ln x\right)\right), \quad x \in (0, \infty)}$$

**MATH 202.10 (Term 152)**

**Quiz 4 (Sects. 4.6 & 4.7)** **Duration: 20min**

**Name:** **ID number:**

**1.)(5pts)Solve the DE:** $y'' - 16y = \frac{e^{4x}}{e^{4x}+e^{-4x}}$.
**2.)(5pts)Solve the DE:** $x^3y''' + 3x^2y'' + xy' - y = 0$.

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1.) We solve $y'' - 16y = 0$.
Its auxiliary equation is $m^2 - 16 = 0$
$m = 4, -4$
$\Rightarrow y_c = c_1 e^{4x} + c_2 e^{-4x}$.
We use variation of parameters to find $y_p$.
We have $y_p = u_1 y_1 + u_2 y_2$,
$y_1 = e^{4x}$, $y_2 = e^{-4x}$, and
$u_1' = \frac{-y_2 f(x)}{W}$, $u_2' = \frac{y_1 f(x)}{W}$,
with $f(x) = \frac{e^{4x}}{e^{4x}+e^{-4x}}$.

$$W = \begin{vmatrix} e^{4x} & e^{-4x} \\ 4e^{4x} & -4e^{-4x} \end{vmatrix} = -8.$$

$$u_1' = -\frac{e^{-4x}}{-8}\frac{e^{4x}}{e^{4x}+e^{-4x}} = \frac{1}{8}\frac{1}{e^{4x}+e^{-4x}} = \frac{e^{4x}}{8(e^{8x}+1)}$$

$$u_1 = \frac{1}{8}\int \frac{e^{4x}}{e^{8x}+1}dx.$$

$w = e^{4x}$, $dw = 4e^{4x}dx$

$$\int \frac{e^{4x}}{e^{8x}+1}dx = \frac{1}{4}\int \frac{dw}{w^2+1} = \frac{1}{4}\tan^{-1} w$$

$\Rightarrow u_1 = \frac{1}{32}\tan^{-1}(e^{4x})$.

$$u_2' = \frac{e^{4x}}{-8}\frac{e^{4x}}{e^{4x}+e^{-4x}} = -\frac{1}{8}\frac{e^{8x}}{e^{4x}+e^{-4x}} = -\frac{1}{8}\frac{e^{12x}}{e^{8x}+1}$$

$$u_2 = -\frac{1}{8}\int \frac{e^{12x}}{e^{8x}+1}dx$$

$w = e^{4x}$, $dw = 4e^{4x}dx$

$$\int \frac{e^{12x}}{e^{8x}+1}dx = \frac{1}{4}\int \frac{w^2}{w^2+1}dw = \frac{1}{4}\int\left(1 - \frac{1}{w^2+1}\right)dw = \frac{1}{4}(w - \tan^{-1} w)$$

$u_2 = -\frac{1}{32}(e^{4x} - \tan^{-1} e^{4x})$.

$\Rightarrow y_p = -\frac{1}{32} + \frac{1}{32}(e^{4x} + e^{-4x})\tan^{-1} e^{4x}$.

$y = y_c + y_p$

2.) $y = x^m$.
The auxiliary eq is
$m(m-1)(m-2) + 3m(m-1) + m - 1 = 0$
$m^3 = 1$
$(m-1)(m^2+m+1) = 0$
$m = 1$, $\Delta = 1 - 4 = -3$
$m = \frac{-1 \pm i\sqrt{3}}{2}$

$$\boxed{y = c_1 x + x^{-\frac{1}{2}}\left(c_2\cos\left(\tfrac{\sqrt{3}}{2}\ln x\right) + c_3\sin\left(\tfrac{\sqrt{3}}{2}\ln x\right)\right) \quad x \in (0, \infty)}$$