King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Math 201
Final Exam
Term 152
Wednesday 11/05/2016
Net Time Allowed: 165 minutes

MASTER VERSION

1. The **length** of the parametric curve

$$x = t^2,\ y = \frac{1}{3}t^3 - t,\ 0 \le t \le \sqrt{3}$$

is equal to

$$L = \int_0^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\sqrt{3}} \sqrt{(2t)^2 + (t^2-1)^2}\, dt$$

$$= \int_0^{\sqrt{3}} \sqrt{4t^2 + t^4 - 2t^2 + 1}\, dt$$

$$= \int_0^{\sqrt{3}} \sqrt{t^4 + 2t^2 + 1}\, dt$$

$$= \int_0^{\sqrt{3}} \sqrt{(t^2+1)^2}\, dt$$

$$= \int_0^{\sqrt{3}} t^2 + 1\, dt = \frac{1}{3}t^3 + t\Big]_0^{\sqrt{3}}$$

$$= \frac{1}{3}\cdot 3\sqrt{3} + \sqrt{3} - 0 = \sqrt{3} + \sqrt{3} = 2\sqrt{3}$$

(a) $2\sqrt{3}$

(b) 3

(c) $1 + \sqrt{3}$

(d) $\frac{\sqrt{3}}{3}$

(e) 2

2. The **slope** of the tangent line to the polar curve $r = \cos(2\theta) = f(\theta)$ at the point corresponding to $\theta = \frac{\pi}{4}$ is

$$\text{slope} = \frac{dy}{dx}\Big|_{\theta=\frac{\pi}{4}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}\Big|_{\theta=\frac{\pi}{4}}$$

$$= \frac{-2\sin(2\theta)\sin\theta + \cos(2\theta)\cos\theta}{-2\sin(2\theta)\cos\theta - \cos(2\theta)\sin\theta}\Big|_{\theta=\frac{\pi}{4}}$$

$$= \frac{-2\cdot\frac{\sqrt{2}}{2} + 0}{-2\cdot\frac{\sqrt{2}}{2} - 0}$$

$$= \underline{1}$$

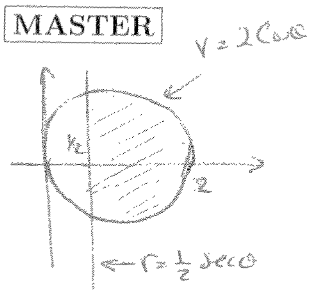
(a) 1

(b) $\sqrt{2}$

(c) 0

(d) $\frac{1}{2}$

(e) -1



3. The **area** of the region inside the curve $r = 2\cos\theta$ and to the right ~~by~~ of the vertical line $r = \dfrac{1}{2}\sec\theta$ is equal to

(a) $\dfrac{2\pi}{3} + \dfrac{\sqrt{3}}{4}$

(b) $\pi - \sqrt{3}$

(c) $2\pi + \sqrt{3}$

(d) $\dfrac{2\pi}{3} - \sqrt{3}$

(e) $\dfrac{\pi}{3} + \dfrac{\sqrt{3}}{2}$

• pts of Intersection

$2\cos\theta = \frac{1}{2}\sec\theta \Rightarrow \cos^2\theta = \frac{1}{4}$

$\Rightarrow \cos\theta = \pm\frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

. By Symmetry,

$A = 2\int_0^{\pi/3} \frac{1}{2}\left(4\cos^2\theta - \frac{1}{4}\sec^2\theta\right) d\theta$

$= \int_0^{\pi/3} 2(1+\cos(2\theta)) - \frac{1}{4}\sec^2\theta \; d\theta$

$= 2\theta + \sin(2\theta) - \frac{1}{4}\tan\theta \Big]_0^{\pi/3}$

$= \frac{2\pi}{3} + \frac{\sqrt{3}}{2} - \frac{1}{4}\sqrt{3} - 0$

$= \frac{2\pi}{3} + \frac{\sqrt{3}}{4}$

4. The **volume** of the box (parallelpiped) determined by the vectors $\vec{u} = <1, 2, -1>$, $\vec{v} = <0, -3, 1>$, $\vec{w} = <2, 0, 3>$ is equal to

(a) 11

(b) -11

(c) 13

(d) 9

(e) 15

$(u \times v)\cdot w = \begin{vmatrix} 1 & 2 & -1 \\ 0 & -3 & 1 \\ 2 & 0 & 3 \end{vmatrix}$

$= 1(-9-0) - 2(0-2) - 1(0+6)$

$= -9 + 4 - 6$

$= -11$

$\text{Volume} = |(u\times v)\cdot w| = |-11| = 11$

5. If the point $(k, 3, -5)$ lies on the plane containing the points $\underset{A}{(0,0,0)}, \underset{B}{(2,0,-1)}, \underset{C}{(2,-1,0)}$, then $k =$

(a) 4

(b) -2

(c) 6

(d) 3

(e) 0

$\overrightarrow{AB} = \langle 2, 0, -1 \rangle$; $\overrightarrow{AC} = \langle 2, -1, 0 \rangle$

$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 2 & -1 & 0 \end{vmatrix} = \langle -1, -2, -2 \rangle$

$\vec{n}$ with $A = (0,0,0)$: Eq. of the plane is

$-(x-0) - 2(y-0) - 2(z-0) = 0$

$\boxed{x + 2y + 2z = 0}$

the point $(k, 3, -5)$ lies on the plane $\Longrightarrow$

$k + 6 - 10 = 0$

$\Rightarrow k = 4$

6. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{\sqrt{2x - y^2} - y}{x^2y - y^5} = \lim \frac{\sqrt{2x-y^2} - y}{y(x^2 - y^4)} \cdot \frac{\sqrt{2x-y^2} + y}{\sqrt{2x-y^2} + y}$

(a) $\dfrac{1}{2}$

(b) 1

(c) 3

(d) -2

(e) does not exist.

$= \lim \dfrac{(2x - y^2) - y^2}{y(x - y^2)(x+y^2) \cdot [\sqrt{2x-y^2} + y)}$

$= \lim \dfrac{2(x - y^2)}{y(x-y^2)(x+y^2)(\sqrt{2x-y^2} + y)}$

$= \displaystyle\lim_{(x,y)\to(1,1)} \frac{2}{y(x+y^2)(\sqrt{2x-y^2} + y)}$

$= \dfrac{2}{1(2)(2)} = \dfrac{1}{2}$

7. Where is $f(x,y) = \dfrac{y\sqrt{x}}{y-x^2}$ **continuous**?

$x \geq 0$

$y - x^2 \neq 0$

(a) $\{(x,y) : x \geq 0,\ y - x^2 \neq 0\}$

(b) $\{(x,y) : x \geq 0,\ y \geq 0,\ y - x^2 \neq 0\}$

(c) $\{(x,y) : xy \geq 0,\ y - x^2 \neq 0\}$

(d) $\{(x,y) : xy \geq 0,\ y - x^2 > 0\}$

(e) $\{(x,y) : x \geq 0,\ y - x^2 > 0\}$.

8. If $f(x,y,z) = \ln(e^{yz} + xz)$, then $f_{xz}(1,0,2) =$

$$f_x(x,y,z) = \frac{1}{e^{yz}+xz} \cdot z = \frac{z}{e^{yz}+xz}$$

$$f_{xz}(x,y,z) = \frac{(e^{yz}+xz)\cdot 1 - z\,(e^{yz}\cdot y + x)}{(e^{yz}+xz)^2}$$

$$f_{xz}(1,0,2) = \frac{(1+2) - 2\,(0+1)}{(1+2)^2}$$

$$= \frac{3-2}{9} = \frac{1}{9}$$

(a) $\dfrac{1}{9}$

(b) $\dfrac{2}{3}$

(c) $\dfrac{3}{4}$

(d) $-\dfrac{1}{3}$

(e) 2

9. Let $f(x,y,z) = 3x^2 + 2y^2 - 4z$, $P(-1,-3,2)$, and $Q(-4,1,-2)$. The **directional derivative** of f at P in the direction of $\overrightarrow{PQ}$ is equal to

$\nabla f = \langle 6x, 4y, -4 \rangle$

$\nabla f(-1,-3,2) = \langle -6, -12, -4 \rangle$

$\overrightarrow{PQ} = \langle -3, 4, -4 \rangle$, $\|\overrightarrow{PQ}\| = \sqrt{9+16+16} = \sqrt{41}$

$\vec{u} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{1}{\sqrt{41}} \langle -3, 4, -4 \rangle$

$D_u f \big|_P = \nabla f|_P \cdot \vec{u}$

$= \langle -6, -12, -4 \rangle \cdot \frac{1}{\sqrt{41}} \langle -3, 4, -4 \rangle$

$= \frac{1}{\sqrt{41}} (18 - 48 + 16)$

$= -\frac{14}{\sqrt{41}}$

(a) $-\frac{14}{\sqrt{41}}$

(b) 0

(c) -1

(d) $\sqrt{41}$

(e) $\frac{-10}{\sqrt{41}}$

10. An equation for the **tangent plane** to the surface $z = \sqrt{2y - x^3}$ at the point $(0,2,2)$ is

(a) $y - 2z = -2$

(b) $z = x + y$

(c) $2x + y - 3z = -4$

(d) $z = 2y - 2$

(e) $x - 3y + 2z = -2$

$f(x,y) = \sqrt{2y - x^3}$

$f_x(x,y) = \frac{-3x^2}{2\sqrt{2y - x^3}} \Longrightarrow f_x(0,2) = 0$

$f_y(x,y) = \frac{2}{2\sqrt{2y - x^3}} \Longrightarrow f_y(0,2) = \frac{1}{2}$

Eq. of tangent plane:

$z = f(0,2) + f_x(0,2)(x-0) + f_y(0,2)(y-2)$

$z = 2 + 0 + \frac{1}{2}(y-2)$

$z = 2 + \frac{1}{2}y - 1$

$z = 1 + \frac{1}{2}y$

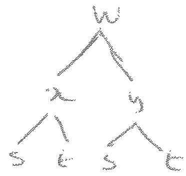
$2z = 2 + y$

$y - 2z = -2$

11. If $w = f\left(ts^2, \frac{s}{t}\right)$, $\frac{\partial f}{\partial x}(x,y) = xy$, $\frac{\partial f}{\partial y}(x,y) = \frac{x^2}{3}$, then $\frac{\partial w}{\partial t} =$

Let $x = ts^2$, $y = \frac{s}{t}$

$w = f(x,y)$, $x = ts^2$, $y = \frac{s}{t}$



(a) $\frac{2}{3}s^5$

(b) 0

(c) $\frac{t^2 s^3}{3}$

(d) $\frac{-t^2}{3}$

(e) $\frac{s^5}{3}$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$= (xy)(s^2) + \left(\frac{x^2}{3}\right)\cdot\left(-\frac{s}{t^2}\right)$$

$$= (s^3)(s^2) + \frac{t^2 s^4}{3}\cdot -\frac{s}{t^2}$$

$$= s^5 - \frac{1}{3}s^5$$

$$= \frac{2}{3}s^5$$

12. The function $f(x,y) = 1 + x^2 - y + \ln(x+y)$ has one critical point (a,b) and

(a) f has a saddle point at (a,b)

(b) f has a local minimum at (a,b)

(c) f has a local maximum at (a,b)

(d) $f(a,b) = \frac{5}{2}$

(e) $f_{xx}(a,b) = 0$

$$\begin{cases} f_x = 2x + \frac{1}{x+y} = 0 & \sim (1) \\ f_y = -1 + \frac{1}{x+y} = 0 & \sim (2) \end{cases}$$

Diff: $2x + 1 = 0$

$\Rightarrow x = -\frac{1}{2}$

$\xrightarrow{(2)} 1 = \frac{1}{x+y} \Rightarrow x+y = 1 \Rightarrow y = \frac{3}{2}$

the critical pt is $\boxed{(a,b) = (-\frac{1}{2}, \frac{3}{2})}$

. $f_{xx} = 2 - \frac{1}{(x+y)^2}$

. $f_{yy} = -\frac{1}{(x+y)^2}$

. $f_{xy} = -\frac{1}{(x+y)^2}$

$D(x,y) = \left(2 - \frac{1}{(x+y)^2}\right)\left(-\frac{1}{(x+y)^2}\right) - \frac{1}{(x+y)^4}$

$D(-\frac{1}{2}, \frac{3}{2}) = (2-1)(-1) - 1 = -2 < 0 \Rightarrow$ Saddle pt

13. The **average value** of $f(x,y) = \dfrac{xy^3}{x^2+1}$ over the rectangular region $R: 0 \le x \le 1,\ 0 \le y \le 2$ is equal to

(a) $\ln 2$

(b) 2

(c) $3\ln 2$

(d) $\dfrac{1}{2}$

(e) $4\ln 2$

Area of the Rect. region = $1 \times 2 = 2$

$$f_{ave} = \frac{1}{2}\iint_R f(x,y)\,dA$$

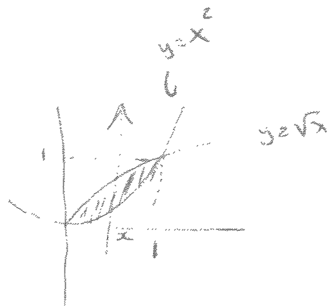
$$= \frac{1}{2}\int_0^1\int_0^2 \frac{xy^3}{x^2+1}\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 \frac{x}{x^2+1}\cdot\frac{1}{4}y^4\Big]_{y=0}^{y=2}\,dx$$

$$= \frac{1}{2}\int_0^1 \frac{x}{x^2+1}\cdot(4-0)\,dx$$

$$= 2\cdot\frac{1}{2}\ln(x^2+1)\Big]_0^1$$

$$= \ln 2 - \ln 1 = \ln 2.$$



14. The value of the double integral

$$\iint_R (x+y)\,dA,$$

where R is the region bounded by the curves $y = \sqrt{x}$ and $y = x^2$, is equal to

(a) $\dfrac{3}{10}$

(b) $\dfrac{1}{4}$

(c) $\dfrac{3}{20}$

(d) $\dfrac{1}{5}$

(e) $\dfrac{2}{3}$

$$\int_0^1\int_{x^2}^{\sqrt{x}} (x+y)\,dy\,dx$$

$$\int_0^1 xy + \frac{1}{2}y^2\Big]_{y=x^2}^{y=\sqrt{x}}\,dx$$

$$\int_0^1 \left(x\sqrt{x} + \frac{1}{2}x\right) - \left(x^3 + \frac{1}{2}x^4\right)\,dx$$

$$\frac{2}{5}x^{5/2} + \frac{1}{4}x^2 - \frac{1}{4}x^4 - \frac{1}{10}x^5\Big]_0^1$$

$$\frac{2}{5} + \frac{1}{4} - \frac{1}{4} - \frac{1}{10}$$

$$\frac{4}{10} - \frac{1}{10} = \frac{3}{10}$$

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Math 201, Final Exam, Term 152
Wednesday May 11, 2016
Time allowed: 165 min

Name: Space KEY
ID # :
Serial #:

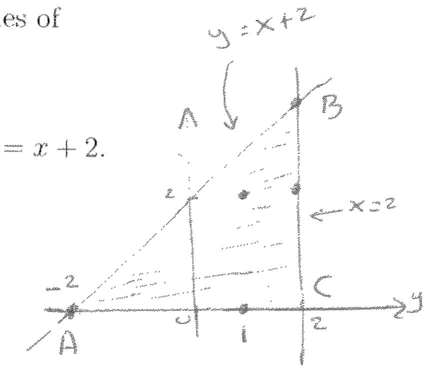
Section #:

Question Number	Marks	Maximum Points
15		14
16		10
17		10
18		10
19		12
20		14
Total		70

15. **(14 points)** Find the absolute maximum and minimum values of

$$f(x,y) = x^2 - y^2 - 2x + 4y$$

on the triangular region bounded by the lines $y = 0$, $x = 2$, $y = x + 2$.



- $\begin{cases} f_x(x,y) = 2x - 2 = 0 \\ f_y(x,y) = -2y + 4 = 0 \end{cases} \Longrightarrow \boxed{(x,y) = (1,2)}$ (2)

- <u>On AB</u>: $y = x+2$, $-2 \le x \le 2$

  $f(x,y) = f(x, x+2) = x^2 - (x+2)^2 - 2x + 4(x+2) = -2x + 4$, $-2 \le x \le 2$

  $f'(x, x+2) = -2$

  So no critical pts in the interior of AB. (3)

  <u>endpoints</u>: $\boxed{(-2,0),\ (2,4)}$

- <u>On BC</u>: $x = 2$, $0 \le y \le 4$

  $f(x,y) = f(2,y) = 4 - y^2 - 4 + 4y = -y^2 + 4y$, $0 \le y \le 4$

  $f'(2,y) = -2y + 4 = 0 \Rightarrow y = 2$ (3)

  <u>points</u>: $\boxed{(2,2),\ (2,0),\ (2,4)}$

- <u>on AC</u>: $y = 0$, $-2 \le x \le 2$

  $f(x,y) = f(x,0) = x^2 - 0 - 2x + 0 = x^2 - 2x$, $-2 \le x \le 2$

  $f'(x,0) = 2x - 2 = 0 \Rightarrow x = 1$ (3)

  <u>points</u>. $\boxed{(1,0),\ (-2,0),\ (2,0)}$

- $f(-2,0) = 8$
  $f(2,4) = 0$
  $f(2,0) = 0$ (1)
  $f(1,2) = 3$
  $f(2,2) = 4$
  $f(1,0) = -1$

- <u>Conclusion</u>: The abs. max. value of f is 8 & it occurs at $(-2,0)$ (1)
  ______ min ______ ___ -1 & ______ $(1,0)$ (1)

16. **(10 points)** Use Lagrange Multipliers to find the extreme values of

$$f(x, y, z) = x - y + z$$

on the unit sphere $x^2 + y^2 + z^2 = 1$

. Set $g(x,y,z) = x^2+y^2+z^2-1$

. We need to solve the system

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g = 0 \end{cases} \Longrightarrow \begin{cases} 1 = \lambda(2x) & \text{——— (1)} \\ -1 = \lambda(2y) & \text{——— (2)} \\ 1 = \lambda(2z) & \text{——— (3)} \\ x^2+y^2+z^2 = 1 & \text{——— (4)} \end{cases}$$

(2)

$$\overset{1,2,3}{\Longrightarrow} \quad x = \frac{1}{2\lambda}, \quad y = -\frac{1}{2\lambda}, \quad z = \frac{1}{2\lambda} \qquad (\lambda \neq 0)$$

(1)

$$\overset{4}{\Longrightarrow} \quad \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 1$$

$$\Longrightarrow \quad 3 = 4\lambda^2$$

$$\Longrightarrow \quad \lambda = \pm\frac{\sqrt{3}}{2}$$

(2)

. $\lambda = \frac{\sqrt{3}}{2} \Longrightarrow (x, y, z) = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ (1)

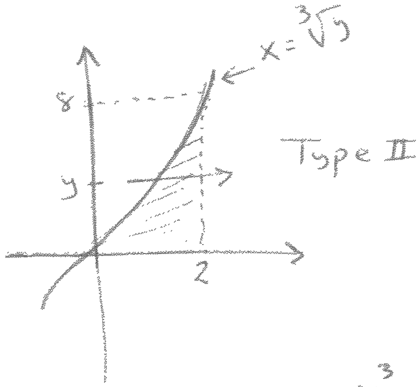
. $\lambda = -\frac{\sqrt{3}}{2} \Longrightarrow (x, y, z) = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ (1)

. $f\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{3}{\sqrt{3}} = \sqrt{3}$ (0.5) $\longleftrightarrow$ max value of f is $\sqrt{3}$ (1)

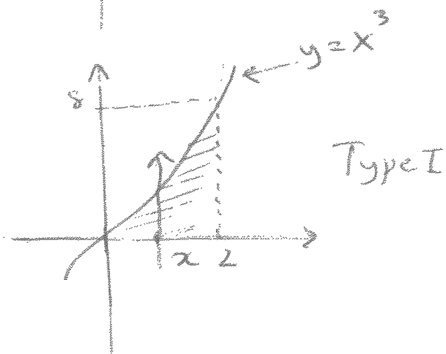
. $f\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = -\frac{3}{\sqrt{3}} = -\sqrt{3}$ (0.5) $\longleftrightarrow$ min value of f is $-\sqrt{3}$ (1)

17. **(10 points)** Evaluate $\int_0^8 \int_{\sqrt[3]{y}}^2 \sin(x^4)\,dx\,dy$

. Reverse the order of integration



$R: \sqrt[3]{y} \le x \le 2 \,, \quad 0 \le y \le 8$



$R: \quad 0 \le x \le 2, \quad 0 \le y \le x^3$

3 3

$$\int_0^8 \int_{\sqrt[3]{y}}^2 \sin(x^4)\,dx\,dy = \int_0^2 \int_0^{x^3} \sin(x^4)\,dy\,dx$$

$$= \int_0^2 \sin(x^4)\cdot y \Big]_{y=0}^{y=x^3} dx \qquad 1$$

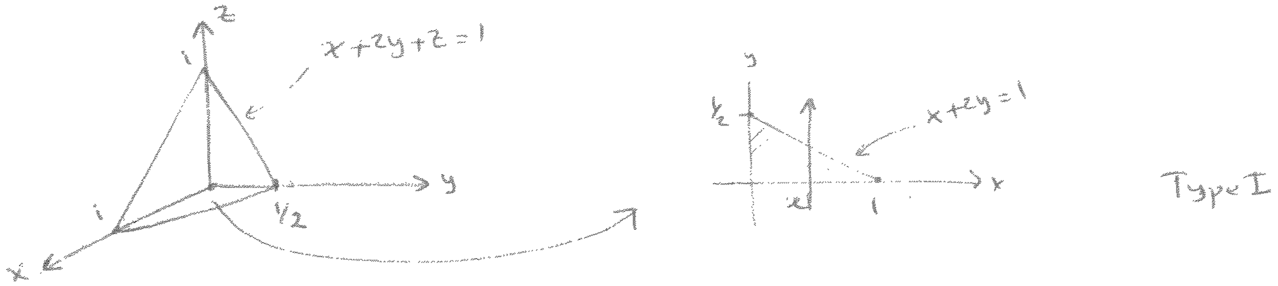
$$= \int_0^2 x^3 \sin(x^4)\,dx \qquad 1$$

$$= -\frac{1}{4}\cos(x^4)\Big]_0^2 \qquad 1$$

$$= -\frac{1}{4}\left(\cos(16) - 1\right) \qquad 1$$

$$\overset{\text{or}}{=} \frac{1}{4}\left(1 - \cos(16)\right).$$

18. **(10 points)** Evaluate $\iiint_D x\,dV$, where D is the solid in the first octant bounded by the coordinate planes and the plane $x + 2y + z = 1$.



$0 \le z \le 1 - x - 2y$

$0 \le y \le \frac{1-x}{2}$; $0 \le x \le 1$

$$\iiint_D x\,dV = \int_0^1 \int_0^{\frac{1-x}{2}} \int_0^{1-x-2y} x \;dz\;dy\;dx$$ (2 +(2 +(2

$$= \int_0^1 \int_0^{\frac{1-x}{2}} x(1-x-2y)\;dy\;dx$$

$$= \int_0^1 \int_0^{\frac{1-x}{2}} x - x^2 - 2xy\;dy\;dx$$ (1)

$$= \int_0^1 (x-x^2)y - xy^2 \Big]_{y=0}^{y=\frac{1-x}{2}} dx$$ (1)

$$= \int_0^1 (x-x^2)\left(\frac{1-x}{2}\right) - x\left(\frac{1-x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int_0^1 x - 2x^2 + x^3\;dx$$ (1)

$$= \frac{1}{4}\cdot \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4\Big]_0^1$$

$$= \frac{1}{4}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$$

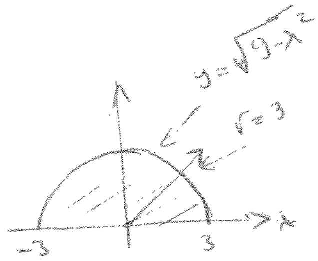
$$= \frac{1}{4}\cdot\frac{1}{12}$$

$$= \frac{1}{48}$$ (1)

19. **(12 points)** Use **cylindrical coordinates** to evaluate the triple integral

$$I = \int_{3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} \sqrt{x^2+y^2}\,dz\,dy\,dx$$

$$D = \{(x,y,z) : -3 \le x \le 3,\ 0 \le y \le \sqrt{9-x^2},\ 0 \le z \le 9-x^2-y^2\}$$



$$= \{(r,\theta,z) : 0 \le r \le 3,\ 0 \le \theta \le \pi,\ 0 \le z \le 9-r^2\}$$

$$I = \int_0^{\pi}\int_0^{3}\int_0^{9-r^2} r\ \ r\,dz\,dr\,d\theta$$

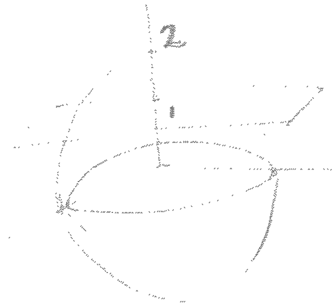
$$= \int_0^{\pi}\int_0^{3} r^2(9-r^2)\,dr\,d\theta$$

$$= \int_0^{\pi} 3r^3 - \frac{1}{5}r^5\Big]_{r=0}^{r=3}\,d\theta$$

$$= \int_0^{\pi} \left(3^4 - \frac{1}{5}3^5\right)d\theta = 3^4\cdot\frac{2}{5}\cdot\theta\Big]_0^{\pi}$$

$$= \frac{162}{5}\pi$$

20. **(14 points)** Use **spherical coordinates** to find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 4$ and above the plane $z = 1$.



- $x^2+y^2+z^2 = 4 \implies \rho^2 = 4 \implies \rho = 2 \quad (\rho \geq 0)$ 1
- $z = 1 \implies \rho\cos\phi = 1 \implies \rho = \sec\phi$ 1
- pts of intersection : $2 = \sec\phi \implies \cos\phi = \frac{1}{2}$
  $\implies \phi = \frac{\pi}{3}$ 1
- $x^2+y^2+z^2 = 4$ & $z = 1 \implies x^2 + y^2 = 3 \implies 0 \leq \theta \leq 2\pi$

$$D = \{(\rho, \phi, \theta) : \sec\phi \leq \rho \leq 2 \ , \ 0 \leq \phi \leq \frac{\pi}{3} \ , \ 0 \leq \theta \leq 2\pi\}$$
1 2 2

$$V = \iiint_D dV$$ 1

$$= \int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$ 1

$$= \int_0^{2\pi}\int_0^{\pi/3} \sin\phi \cdot \frac{1}{3}\rho^3 \Big]_{\rho=\sec\phi}^{\rho=2} d\phi \, d\theta$$ 1

$$= \int_0^{2\pi}\int_0^{\pi/3} \frac{1}{3}\sin\phi \cdot (8 - \sec^3\phi) \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \frac{1}{3}(8\sin\phi - \tan\phi \sec^2\phi) \, d\phi \, d\theta$$ 1

$$= \int_0^{2\pi} \frac{1}{3} \cdot \left(-8\cos\phi - \frac{1}{2}\tan^2\phi\right)\Big]_{\phi=0}^{\phi=\frac{\pi}{3}} d\theta$$

$$= \int_0^{2\pi} \frac{1}{3}\left[-8\left(\frac{1}{2}\right) - \frac{1}{2}(3) + 8\right] d\theta$$

$$= \int_0^{2\pi} \frac{1}{3} \cdot \frac{5}{2} \, d\theta = \frac{5}{6} \cdot \theta\Big]_0^{2\pi}$$ 1

$$= \frac{5\pi}{3}$$ 1