King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 201 - Exam II - Term 152

Duration: 120 minutes

Solutions

Name:______________________ ID Number:______________________

Section Number:______________________ Serial Number:______________________

Class Time:______________________ Instructor's Name:______________________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 7 pages of problems (Total of 7 Problems)

Question Number	Points	Maximum Points
1		20
2		6
3		12
4		18
5		24
6		10
7		10
Total		100

1. Consider the following lines:

$$L_1 : x = 1 + 2t,\ y = 2 + 3t,\ z = 3 + 4t,\ t \in (-\infty, \infty)$$

$$L_2 : x = 2 + s,\ y = 4 + 2s,\ z = -1 - 4s,\ s \in (-\infty, \infty)$$

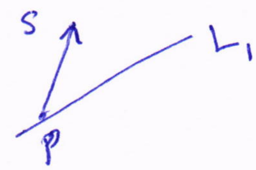
(a)[**7 points**] Find the point of intersection between L_1 and L_2.

$$\begin{cases} x = x \\ y = y \\ z = z \end{cases} \Rightarrow \begin{cases} 1+2t = 2+s \\ 2+3t = 4+2s \\ 3+4t = -1-4s \end{cases} \Rightarrow \begin{cases} 2t - s = 1 & \sim (1) \\ 3t - 2s = 2 & \sim (2) \\ 4t + 4s = -4 & \sim (3) \end{cases} \quad \boxed{2}$$

- Solving the system (1) & (2) gives $t = 0,\ s = -1$ $\boxed{2}$
- substituting $t=0,\ s=-1$ in (3) : $0 - 4 = -4$ (✓) $\boxed{1}$
- Substituting $t=0$, in L_1 (or $s=-1$ in L_2) we get the point of intersection

$$(x, y, z) = (1, 2, 3) \quad \boxed{2}$$

(b)[**7 points**] Find an equation for the plane containing the point $S(2, 0, 1)$ and the line L_1.



$\boxed{1}$. a point on L_1 : $P = (1, 2, 3)$, when $t = 0$

$\boxed{1}$. a vector parallel to L_1 : $\vec{v} = \langle 2, 3, 4 \rangle$

$\boxed{1}$. $\vec{PS} = \langle 2-1, 0-2, 1-3 \rangle = \langle 1, -2, -2 \rangle$

$\boxed{2}$. a normal to the plane $\vec{n} = \vec{PS} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & -2 & -2 \\ 2 & 3 & 4 \end{vmatrix} = -2i - 8j + 7k$

. an equation for the plane $(\vec{n}, S)$:

$\boxed{1}$ $$-2(x-2) - 8(y-0) + 7(z-1) = 0$$

$\boxed{1}$ $$\Rightarrow -2x - 8y + 7z = 3$$

(c)[**6 points**] Find the distance from the point $S(2, 0, 1)$ to the line L_1.

From part (b) :

$$d = \frac{|\vec{PS} \times \vec{v}|}{|v|} \quad \boxed{4} \ \{ \boxed{2} \text{ for the formula} ; \boxed{2} \text{ for finding } \vec{PS} \times \vec{v} \}$$

$$= \frac{\sqrt{4 + 64 + 49}}{\sqrt{4 + 9 + 16}} = \frac{\sqrt{117}}{\sqrt{29}} = \sqrt{\frac{117}{29}} \quad \boxed{2}$$

2. Identify the surface given by each of the following equations (give the name only):
(a) **[3 points]** $2x^2 + 3y^2 - 4z^2 = 0$

$4z^2 = 2x^2 + 3y^2$, an elliptic Cone $\boxed{3}$
(axis : z-axis)

(b) **[3 points]** $z^2 - x^2 - 5y^2 = 4$

$\frac{z^2}{4} - \frac{x^2}{4} - \frac{5y^2}{4} = 1$, a Hyperboloid of two sheets $\boxed{3}$
(axis: z-axis)

3. Find the limit or show that it does not exist.

(a) **[6 points]** $\lim\limits_{(x,y)\to(0,0)} y^2 \sin\left(\frac{2}{xy}\right)$.

$\boxed{2}$ $-1 \leq \sin\left(\frac{2}{xy}\right) \leq 1$, $xy \neq 0$

$\boxed{1}$ $\Rightarrow -y^2 \leq y^2 \sin\left(\frac{2}{xy}\right) \leq y^2$

$\boxed{1}$ Since $\lim\limits_{(x,y)\to(0,0)} y^2 = 0 = \lim\limits_{(x,y)\to(0,0)} -y^2$

$\boxed{2}$ then $\lim\limits_{(x,y)\to(0,0)} y^2 \sin\left(\frac{2}{xy}\right) = 0$ by the Squeeze Theorem.

(b) **[6 points]** $\lim\limits_{(x,y)\to(0,1)} \frac{x^3 + (y-1)^3}{x^3 - (y-1)^2}$.

- along the path $x=0$ (y-axis)

$$\lim_{\substack{(x,y)\to(0,1)\\ \text{along } x=0}} \frac{x^3 + (y-1)^3}{x^3 - (y-1)^2} = \lim_{(x,y)\to(0,1)} \left.\frac{x^3+(y-1)^3}{x^3-(y-1)^2}\right|_{x=0} = \lim_{(x,y)\to(0,1)} -(y-1) = 0$$ $\boxed{2}$

- along the path $y=1$ (a horizontal line)

$$\lim_{\substack{(x,y)\to(0,1)\\ \text{along } y=1}} \frac{x^3+(y-1)^3}{x^3-(y-1)^2} = \lim_{(x,y)\to(0,1)} \left.\frac{x^3+(y-1)^3}{x^3-(y-1)^2}\right|_{y=1} = \lim_{(x,y)\to(0,1)} 1 = 1$$ $\boxed{2}$

- Since the limits along the two paths are not equal, $\boxed{1}$

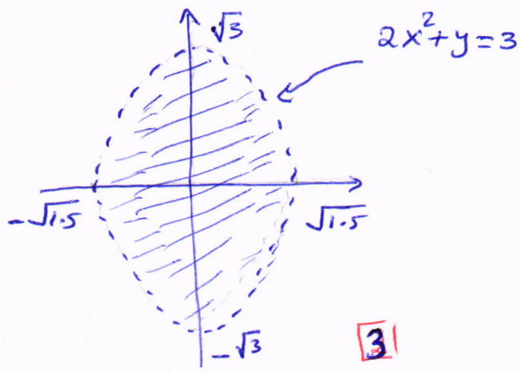
then $\lim\limits_{(x,y)\to(0,1)} \frac{x^3+(y-1)^3}{x^3-(y-1)^2}$ does not exist. $\boxed{1}$

4. Let $f(x, y) = 4\ln(3 - 2x^2 - y^2)$.

(a) [**5 points**] Find and sketch the domain of f.

$$\text{Domain} = \{(x,y) : 3 - 2x^2 - y^2 > 0\} \quad \boxed{2}$$

$$= \{(x,y) : 2x^2 + y^2 < 3\}$$



$\boxed{3}$

(b) [**7 points**] Find the range of f.

For any point (x,y) in the domain of f

$$0 < 3 - 2x^2 - y^2 \le 3$$

$$\Rightarrow \quad -\infty < \ln(3 - 2x^2 - y^2 \le \ln 3$$

$$\Rightarrow \quad -\infty < 4\ln(3 - 2x^2 - y^2 \le 4\ln 3$$

$$\Rightarrow \quad -\infty < f(x,y) \le 4\ln 3$$

So the Range is $(-\infty, 4\ln 3]$.

(c) [**6 points**] Find an equation for the level curve of f passing through the point $(1, 0)$. Sketch the level curve.

The general equation of the level curve is

$$f(x,y) = c \quad , \text{ where } c \text{ is a constant.}$$

Since the level curve passes through $(1,0)$, then

$$f(1,0) = c$$

$$\boxed{2} \quad \Rightarrow \quad c = f(1,0) = 4\ln(3-2-0) = 4\ln 1 = 0$$

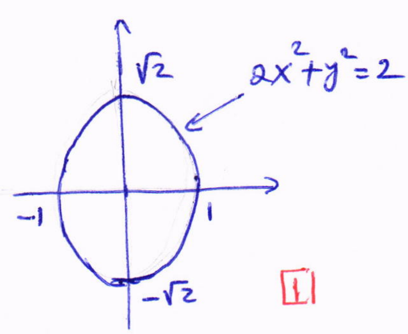
So the required level curve is

$$\boxed{1} \quad 4\ln(3 - 2x^2 - y^2) = 0$$

$$\Rightarrow \quad \ln(3 - 2x^2 - y^2) = 0$$

$$\Rightarrow \quad 3 - 2x^2 - y^2 = 1$$

$$\boxed{2} \quad \Rightarrow \quad 2x^2 + y^2 = 2$$



$\boxed{1}$

5. (a) **[10 points]** Let $z = \tan^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, $x > 0$, $y > 0$. Find

$$(x+y)\left(x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}\right).$$

(Simplify your answer).

$$\frac{\partial z}{\partial x} = \frac{1}{1+\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^2} \cdot \frac{(\sqrt{x}+\sqrt{y})\cdot\frac{1}{2\sqrt{x}} - (\sqrt{x}-\sqrt{y})\cdot\frac{1}{2\sqrt{x}}}{(\sqrt{x}+\sqrt{y})^2} \quad \boxed{2}$$

$$= \frac{\sqrt{y}/\sqrt{x}}{(\sqrt{x}+\sqrt{y})^2+(\sqrt{x}-\sqrt{y})^2} \quad \boxed{1}$$

$$= \frac{\sqrt{y}/\sqrt{x}}{x+2\sqrt{x}\sqrt{y}+y+x-2\sqrt{x}\sqrt{y}+y} = \frac{\sqrt{y}/\sqrt{x}}{2(x+y)} \quad \boxed{1}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^2} \cdot \frac{(\sqrt{x}+\sqrt{y})\cdot\frac{-1}{2\sqrt{y}} - (\sqrt{x}-\sqrt{y})\cdot\frac{1}{2\sqrt{y}}}{(\sqrt{x}+\sqrt{y})^2} \quad \boxed{2}$$

$$= \frac{-\sqrt{x}/\sqrt{y}}{(\sqrt{x}+\sqrt{y})^2+(\sqrt{x}-\sqrt{y})^2}$$

$$= \frac{-\sqrt{x}/\sqrt{y}}{x+2\sqrt{x}\sqrt{y}+y+x-2\sqrt{x}\sqrt{y}+y} = \frac{-\sqrt{x}/\sqrt{y}}{2(x+y)} \quad \boxed{1}$$

$$(x+y)\left(x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}\right) = (x+y)\cdot\left(x\cdot\frac{\sqrt{y}/\sqrt{x}}{2(x+y)} - y\cdot\frac{-\sqrt{x}/\sqrt{y}}{2(x+y)}\right)$$

$$= \frac{1}{2}\left(x\frac{\sqrt{y}}{\sqrt{x}} + y\frac{\sqrt{x}}{\sqrt{y}}\right) \quad \boxed{1}$$

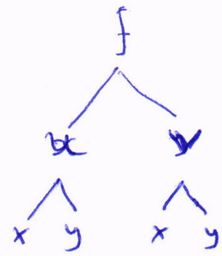
$$= \frac{1}{2}\left(\sqrt{x}\sqrt{y} + \sqrt{y}\sqrt{x}\right) \quad \boxed{1}$$

$$= \frac{1}{2}\cdot 2\sqrt{x}\sqrt{y}$$

$$= \sqrt{xy} \quad \boxed{1}$$

(b)[**6 points**] Let $f(x, y) = \ln(u^3 + v^4)$, $u = 2x + y$, $v = 3x - y$. Find $f_y(1, 1)$.

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} \qquad \boxed{2}$$



$$= \frac{3u^2}{u^3+v^4}\cdot(1) + \frac{4v^3}{u^3+v^4}\cdot(-1) \qquad \boxed{2}$$

$$= \frac{3u^2 - 4v^3}{u^3+v^4}$$

$$\boxed{x=1,\ y=1 \Rightarrow u=3,\ v=2} \qquad \boxed{1}$$

$$\Longrightarrow f_y(1,1) = \frac{3(3)^2 - 4(2)^3}{3^3+2^4}$$

$$= \frac{27-32}{27+16} = \frac{-\cancel{1}5}{43} \qquad \boxed{1}$$

(c)[**8 points**] Find $\dfrac{\partial^2 z}{\partial x\,\partial y}$ if the equation $z^3 - xz - y = 0$ defines z implicitly as a function of x and y.

Let $F(x,y,z) = z^3 - xz - y$

- $$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \qquad \boxed{1}$$
$$= -\frac{-1}{3z^2-x} = \frac{1}{3z^2-x} \qquad \boxed{1}$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \boxed{1}$$
$$= -\frac{-z}{3z^2-x} = \frac{z}{3z^2-x} \qquad \boxed{1}$$

- $$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{1}{3z^2-x}\right)$$
$$= \frac{-\left(6z\cdot\frac{\partial z}{\partial x} - 1\right)}{(3z^2-x)^2} \qquad \boxed{2}$$
$$= -\frac{6z\cdot\frac{z}{3z^2-x} - 1}{(3z^2-x)^2} = -\frac{6z^2-(3z^2-x)}{(3z^2-x)^3}$$
$$= -\frac{3z^2+x}{(3z^2-x)^3} \qquad \boxed{2}$$

6. **[10 points]** Let $f(x, y) = x^3 - x^2y - y^2$. Find the unit vectors $\vec{u}$ such that $D_{\vec{u}}f(1, -1) = -1$.

- We need to find vectors $\vec{u} = \langle u_1, u_2 \rangle$ such that

$$(*) \quad \begin{cases} D_{\vec{u}} f(1,-1) = -1 \\ |\vec{u}| = 1 \end{cases}$$

- $\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 3x^2 - 2xy, -x^2 - 2y \rangle$ $\boxed{1}$

$\nabla f(1,-1) = \langle 5, 1 \rangle$ $\boxed{1}$

$D_{\vec{u}} f(1,-1) = \nabla f(1,-1) \cdot \vec{u} = 5u_1 + u_2$ $\boxed{2}$

- Now the system $(*)$ becomes

$$\begin{cases} 5u_1 + u_2 = -1 & \text{—(1)} \\ u_1^2 + u_2^2 = 1 & \text{—(2)} \end{cases} \quad \boxed{2}$$

$$(1) \Rightarrow u_2 = -1 - 5u_1 \quad \text{—(3)}$$

$$\overset{(2)}{\Longrightarrow} u_1^2 + (-1-5u_1)^2 = 1$$

$$\Rightarrow u_1^2 + 1 + 10u_1 + 25u_1^2 = 1$$

$$\Rightarrow 26u_1^2 + 10u_1 = 0$$

$$\Rightarrow 2u_1(13u_1 + 5) = 0$$

$$\Rightarrow u_1 = 0, \quad u_1 = -\frac{5}{13}$$

- $u_1 = 0 \overset{(3)}{\Longrightarrow} u_2 = -1$
- $u_1 = -\frac{5}{13} \overset{(3)}{\Longrightarrow} u_2 = -1 + \frac{25}{13} = \frac{12}{13}$

- The required unit vectors are

$\vec{u} = \langle 0, -1 \rangle$ $\boxed{2}$

$\vec{u} = \langle -\frac{5}{13}, \frac{12}{13} \rangle$ $\boxed{2}$

7. Let $f(x, y, z) = \dfrac{\sin(xy)}{z}$.

(a) [**7 points**] Find the linearization of f at the point $(2, 0, 1)$.

$L(x,y,z) = f(2,0,1) + f_x(2,0,1)(x-2) + f_y(2,0,1)(y-0) + f_z(2,0,1)(z-1)$ $\boxed{2}$

- $f(2,0,1) = 0$ $\boxed{1}$
- $f_x(x,y,z) = \dfrac{\cos(xy)\cdot y}{z} \Rightarrow f_x(2,0,1) = 0$ $\boxed{1}$
- $f_y(x,y,z) = \dfrac{\cos(xy)\cdot x}{z} \Rightarrow f_y(2,0,1) = 2$ $\boxed{1}$
- $f_z(x,y,z) = -\dfrac{\sin(xy)}{z^2} \Rightarrow f_z(2,0,1) = 0$ $\boxed{1}$

So $L(x,y,z) = 2y$ $\boxed{1}$

(b)[**3 points**] Estimate the value of $f(1.9, 0.1, 1.1)$.

From part (a),

$f(x,y,z) \approx L(x,y,z)$ when (x,y,z) is near $(2,0,1)$ $\boxed{1}$

Since $(1.9, 0.1, 1.1)$ is near $(2,0,1)$, then

$$\begin{aligned} f(1.9, 0.1, 1.1) &\approx L(1.9, 0.1, 1.1) \quad \boxed{1} \\ &= 2(0.1) \\ &= 0.2 \quad \boxed{1} \end{aligned}$$