King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 201 - Exam I - Term 152

Duration: 120 minutes

Solutions

Name:____________________ ID Number:____________________

Section Number:________________ Serial Number:____________________

Class Time: ____________________ Instructor's Name:____________________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 7 pages of problems (Total of 8 Problems)

Question Number	**Points**	**Maximum Points**
1		7
2		15
3		10
4		18
5		10
6		10
7		20
8		10
Total		100

1. **[7 points]** Sketch the parametric curve described by the parametric equations

$$x = \ln t,\ y = \sqrt{t},\ t \geq 1$$

Indicate with an arrow the direction in which the curve is traced as t increases.

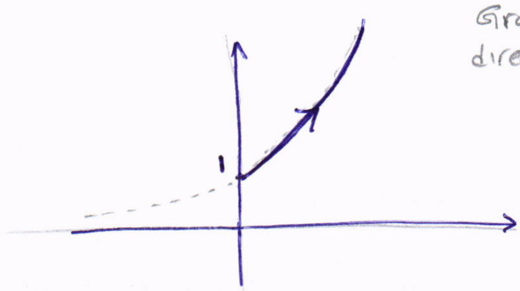
Change to a Cartesian equation:

Graph: 2
direction: 1

$x = \ln t \Rightarrow t = e^{x}$ 1

$y = \sqrt{t} \Rightarrow y = \sqrt{e^{x}} = y = e^{x/2}$ 2

$t \geq 1 \Rightarrow x = \ln t \geq 0$ 1



The part of the curve $y = e^{x/2}$ in the first quadrant.

2. **[5+5+5=15 points]** Consider the parametric curve described by the parametric equations

$$x = \cos^3 t,\ y = \sin^3 t,\ -\infty < t < \infty.$$

(a) Find $\frac{dy}{dx}$. (Simplify your answer)

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\sin^2 t \cdot \cos t}{3\cos^2 t \cdot (-\sin t)} = -\tan t$$

2 2 1

(b) Find an equation for the tangent line to the curve at the point corresponding to $t = \frac{\pi}{4}$.

1 . point : $t = \frac{\pi}{4} \Rightarrow (x, y) = \left(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$

2 . slope $= \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = -\tan\left(\frac{\pi}{4}\right) = -1$ (Using part (a))

. Eq : $y - \frac{1}{2\sqrt{2}} = -\left(x - \frac{1}{2\sqrt{2}}\right) \Rightarrow y = -x + \frac{1}{\sqrt{2}}$

1 1

(c) Find $\frac{d^2y}{dx^2}$. (Simplify your answer)

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx}, \quad y' = \frac{dy}{dx} = -\tan t \text{ (From part (a))}$$

$$= \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{-\sec^2 t}{3\cos^2 t\,(-\sin t)} = \frac{1}{3}\sec^4 t \csc t$$

2 2 1

3. **[10 points]** Find the area of the surface generated by revolving the parametric curve

$$x = \frac{1}{2}t^2,\ y = \frac{1}{3}(2t+1)^{3/2},\ 0 \le t \le 2.$$

about the y-axis.

$$S = \int_0^2 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad 2$$

1 $\frac{dx}{dt} = t$, $\frac{dy}{dt} = \frac{1}{3} \cdot \frac{3}{2} (2t+1)^{1/2} \cdot 2 = (2t+1)^{1/2}$

1 $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = t^2 + 2t + 1 = (t+1)^2$

2 $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = |t+1| = t+1$, $0 \le t \le 2$

$$= \int_0^2 2\pi \cdot \frac{1}{2} t^2 \cdot (t+1)\, dt$$

$$= \pi \int_0^2 (t^3 + t^2)\, dt \quad 1$$

$$= \pi \cdot \left. \frac{1}{4}t^4 + \frac{1}{3}t^3 \right]_0^2 \quad 1$$

$$= \pi \cdot \left(4 + \frac{8}{3}\right)$$

$$= \frac{20\pi}{3} \quad 2$$

4. **[6+6+6=18 points]**

(a) Replace the polar equation $r = 2\tan\theta\sec\theta$ with an equivalent Cartesian equation.

$$r = 2\,\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}$$

$$\Rightarrow r\cos^2\theta = 2\sin\theta \quad 2$$

$$\Rightarrow r^2\cos^2\theta = 2r\sin\theta \quad 2$$

$$\Rightarrow x^2 = 2y \quad 2$$

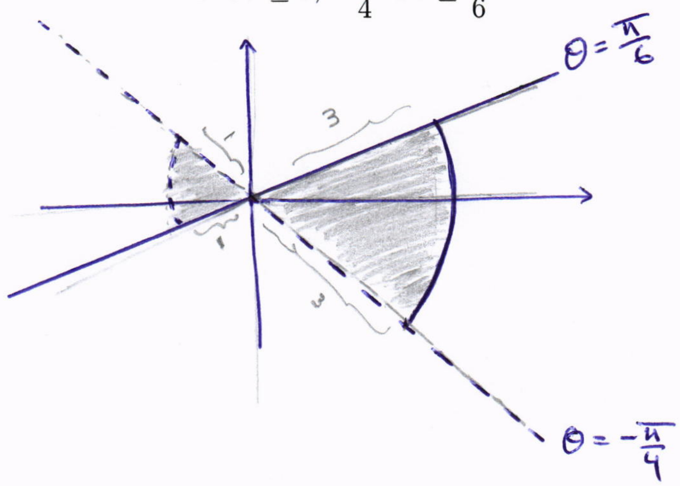
or

$$\Rightarrow y = \frac{1}{2}x^2$$

(b) Graph the set of the polar points (r, θ) such that

$$-1 < r \leq 3,\ -\frac{\pi}{4} < \theta \leq \frac{\pi}{6}$$

3 pts for each sector



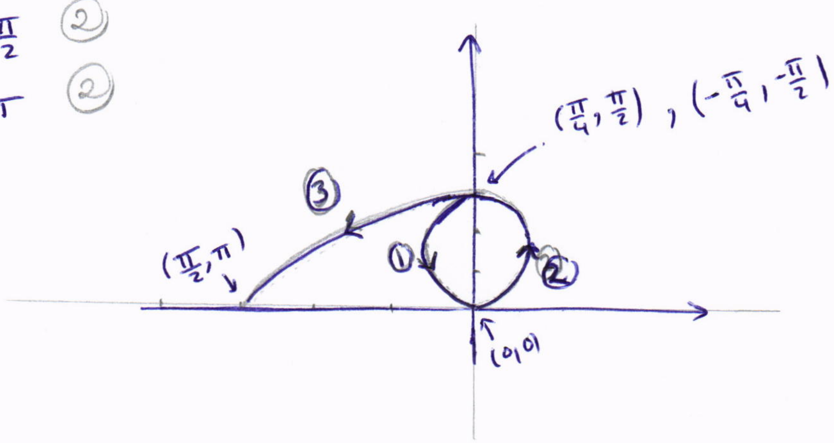
(c) Sketch the graph of the polar equation

$$r = \frac{\theta}{2},\ -\frac{\pi}{2} \leq \theta \leq \pi.$$

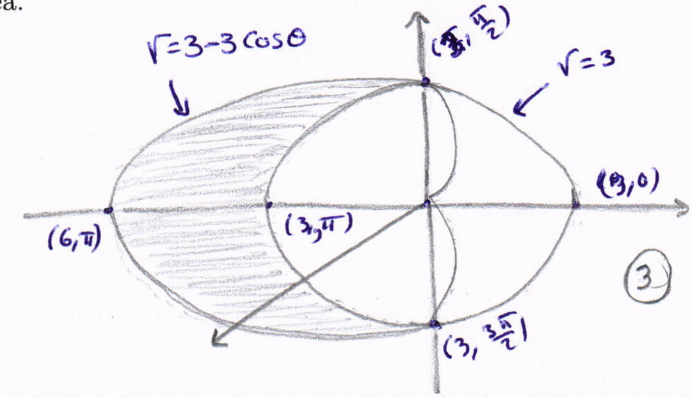
① $-\frac{\pi}{2} \leq \theta \leq 0$ 2

② $0 \leq \theta \leq \frac{\pi}{2}$ 2

③ $\frac{\pi}{2} \leq \theta \leq \pi$ 2



5. **[10 points]** Let R be the region outside the circle $r = 3$ and inside the cardioid $r = 3 - 3\cos\theta$. Sketch R and find its area.



by Symmetry about the polar axis.

- pts of intersection:

$$3-3\cos\theta = 3 \Rightarrow -3\cos\theta = 0 \Rightarrow \cos\theta = 0$$

$$\Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2} \quad 1$$

- $$A = \frac{1}{2}\int_{\pi/2}^{3\pi/2} \left[(3-3\cos\theta)^2 - (3)^2\right] d\theta$$

or 3

$$\overset{\text{or}}{=} 2\cdot\frac{1}{2}\int_{\pi/2}^{\pi} \left[(3-3\cos\theta)^2 - (3)^2\right] d\theta$$

$$= \int_{\pi/2}^{\pi} -18\cos\theta + 9\cos^2\theta \; d\theta$$

$$= 9\int_{\pi/2}^{\pi} -2\cos\theta + \frac{1+\cos(2\theta)}{2} \; d\theta \quad 1$$

$$= 9\left[-2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right]_{\pi/2}^{\pi} \quad 1$$

$$= 9\left[\left(\frac{\pi}{2}\right) - \left(-2 + \frac{\pi}{4}\right)\right]$$

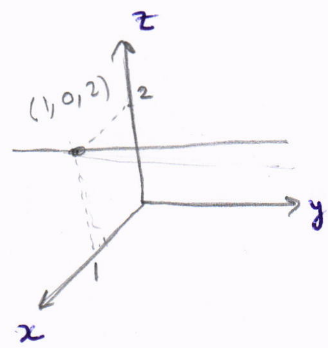
$$= 9\left(\frac{\pi}{4} + 2\right) \quad 1$$

6. **[4+6=10 points]**
(a) Give a geometric description of the set of points (x, y, z) in the $3D-$ space whose coordinates satisfy $x = 1$, $z = 2$.

$\{(x,y,z) : x=1,\ z=2,\ y \text{ is any real number}\}$

1 2

a line parallel to the y-axis and passing through the point $(1,0,2)$ 1



(b) Find the center and the radius of the sphere

$$4x^2 + 4y^2 + 4z^2 + 12x - 16z + 1 = 0.$$

$$x^2 + y^2 + z^2 + 3x - 4z + \frac{1}{4} = 0$$

$$\left(x^2 + 3x + \frac{9}{4}\right) + y^2 + (z^2 - 4z + 4) = -\frac{1}{4} + \frac{9}{4} + 4$$

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z-2)^2 = 6 \quad 2$$

Center : $\left(-\frac{3}{2}, 0, 2\right)$ 2

radius $= \sqrt{6}$ 2

7. **[4+6+5+5=20 points]** Let $P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,3)$ be three points in the 3D-space.

(a) Find $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

$$\overrightarrow{PQ} = \langle 2-1, 1-(-1), -1-0\rangle = \langle 1, 2, -1\rangle \quad 2$$

$$\overrightarrow{PR} = \langle -1-1, 1-(-1), 3-0\rangle = \langle -2, 2, 3\rangle \quad 2$$

(b) Find the vector projection of $\overrightarrow{PQ}$ onto $\overrightarrow{PR}$.

$$2 \quad \text{proj}_{\overrightarrow{PR}} \overrightarrow{PQ} = \left(\frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{|\overrightarrow{PR}|^2}\right)\overrightarrow{PR}$$

$$: \overrightarrow{PQ}\cdot\overrightarrow{PR} = -2+4-3 = -1 \quad 1$$

$$|PR|^2 = 4+4+9 = 17 \quad 1$$

$$= \frac{-1}{17}\langle -2, 2, 3\rangle$$

$$\overset{\text{or}}{=} \left\langle \frac{2}{17}, \frac{-2}{17}, \frac{-3}{17}\right\rangle \quad 2$$

(c) Find the area of the triangle with vertices P, Q and R.

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 3 \end{vmatrix} = 8i - j + 6k \quad 2$$

$$\text{area} = \frac{1}{2}|\overrightarrow{PQ}\times\overrightarrow{PR}| \quad 2$$

$$= \frac{1}{2}\sqrt{64+1+36} = \frac{1}{2}\sqrt{101} \quad 1$$

(d) Find a vector of length 3 that is perpendicular to the plane containing the points P, Q and R.

we can take the vector

$$3\cdot\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{|\overrightarrow{PQ}\times\overrightarrow{PR}|} \quad 3$$

$$= 3\cdot\frac{\langle 8, -1, 6\rangle}{\sqrt{101}}$$ from part (c)

$$= \frac{3}{\sqrt{101}}\langle 8, -1, 6\rangle \quad 2$$

$$\overset{\text{or}}{=} \left\langle \frac{24}{\sqrt{101}}, \frac{-3}{\sqrt{101}}, \frac{18}{\sqrt{101}}\right\rangle.$$

8. **[10 points]** Let $\vec{u}$ and $\vec{v}$ be two vectors in the 3D-space that satisfy

$$\vec{u} + 2\vec{v} = <5, 3, -4> \quad \text{(1)}$$

$$3\vec{u} - \vec{v} = <1, 2, 2>. \quad \text{(2)}$$

Find the angle between $\vec{u}$ and $\vec{v}$.

. Multiply the second equation by 2:

$$\vec{u} + 2\vec{v} = <5, 3, -4>$$
$$6\vec{u} - 2\vec{v} = <2, 4, 4>$$

Adding, we get

$$\cancel{7}\vec{u} = <7, 7, 0>$$

and so $\boxed{\vec{u} = <1, 1, 0>}$ 3

Substitute $\vec{u}$ in (2) to get

$$<3, 3, 0> - \vec{v} = <1, 2, 2>$$

and so $\boxed{\vec{v} = <2, 1, -2>}$ 2

. If θ is the angle between $\vec{u}$ and $\vec{v}$, then

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|} \quad 1$$

$$= \frac{2+1+0}{\sqrt{2}\,\sqrt{9}} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} \quad 2$$

$$\Rightarrow \theta = \frac{\pi}{4} \quad (\text{as } 0 \le \theta \le \pi)$$

2