King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 102 - Term 152 - Exam II

Duration: 120 minutes

KEY

Name:______________ ID Number:______________

Section Number:______________ Serial Number:______________

ClassTime: ______________ Instructor's Name:______________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write neatly and eligibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 7 pages of questions (Total of 10 questions)

Question Number	Points	Maximum Points
1		8
2		8
3		6
4		8
5		14
6		14
7		12
8		10
9		10
10		10
Total		100

1. (8-points) Find the average value of $f(x) = \dfrac{\sin(\pi/x)}{x^2}$ on the interval $[1, 3]$.

$$f_{ave} = \frac{1}{3-1} \int_1^3 \frac{\sin(\pi/x)}{x^2}\, dx$$

Do substitution $u = \frac{\pi}{x}$ $du = -\frac{\pi}{x^2}\, dx$ $x=1 \Rightarrow u=\pi$ $x=3 \Rightarrow u=\pi/3$

$$f_{ave} = \frac{1}{2\pi} \int_{\pi/3}^{\pi} \sin(u)\, du$$

$$= \frac{-1}{2\pi} \cos(u) \Big|_{\pi/3}^{\pi} = \frac{-1}{2\pi} \left(-1 - \frac{1}{2}\right) = \frac{3}{4\pi}$$

2. (8-points) Write out, **without evaluating the coefficients**, the form of the partial fraction decomposition of the function

$$\frac{x^7 + x + 1}{(x+1)^2(x^2+1)(x^4-1)}.$$

$$(x+1)^2(x^2+1)(x^4-1) = (x-1)\ (x+1)^3(x^2+1)^2$$

$$\frac{x^7+x+1}{(x+1)^2(x^2+1)(x^4-1)} = \frac{A_1}{x-1} + \frac{A_2}{x+1} + \frac{A_3}{(x+1)^2} + \frac{A_4}{(x+1)^3} + \frac{A_5 x + A_6}{x^2+1} + \frac{A_7 x + A_8}{(x^2+1)^2}$$

3. (6-points) Evaluate the integral $\int x \sec^2 x dx$.

Using integration by parts with $u = x$ $\quad v = \tan x$

$du = dx \quad dv = \sec^2 x\, dx$

$$\int x \sec^2 x\, dx = x \tan x - \int \tan x\, dx = x \tan x - \ln|\sec x| + c.$$

4. (8-points) Evaluate the integral $\int_0^{\pi/2} \sin^2 x \cos 3x dx$.

Using the trigonometric identity $\sin^2 x = \dfrac{1-\cos 2x}{2}$

$$\int_0^{\pi/2} \sin^2 x \cos 3x\, dx = \int_0^{\pi/2} \frac{\cos 3x}{2} - \frac{\cos 3x \cos 2x}{2}\, dx$$

Using the trigonometric identity $\cos 3x \sin 2x = \frac{1}{2}\left[\cos 5x + \cos x\right]$

$$\int_0^{\pi/2} \sin^2 x \cos 3x\, dx = \int_0^{\pi/2} \frac{\cos 3x}{2} - \frac{\cos 5x}{4} - \frac{\cos x}{4}\, dx$$

$$= \frac{\sin 3x}{6}\Big|_0^{\pi/2} - \frac{\sin 5x}{20}\Big|_0^{\pi/2} - \frac{\sin x}{4}\Big|_0^{\pi/2}$$

$$= \frac{-1}{6} - \frac{1}{20} - \frac{1}{4} = \frac{-7}{15}$$

5. (14-points) Evaluate the integral $\displaystyle\int \frac{t}{\sqrt{t^2-6t+13}}\,dt$.

Let $\displaystyle I = \int \frac{t}{\sqrt{t^2-6t+13}}\,dt$.

$$I = \int \frac{t}{\sqrt{(t-3)^2+4}}\,dt$$

After substitution $u = t-3$, $\displaystyle I = \int \frac{u+3}{\sqrt{u^2+4}}\,du$

Do the trigonometric substitution $u = 2\tan\theta$, $-\pi/2 < \theta < \pi/2$

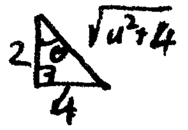
$du = 2\sec^2\theta\, d\theta$

$$I = \int \frac{2\tan\theta+3}{\sqrt{4\tan^2\theta+4}}\, 2\sec^2\theta\, d\theta = \int \frac{2\tan\theta+3}{2\sec\theta}\, 2\sec^2\theta\, d\theta$$

$$= \int (2\tan\theta+3)\sec\theta\, d\theta = 2\int \tan\theta\sec\theta + 3\int \sec\theta\, d\theta$$

$$= 2\sec\theta + 3\ln|\sec\theta+\tan\theta| + c$$

Substitute back the initial variable using



$\sec\theta = \frac{1}{2}\sqrt{u^2+4}$

$$I = 2\cdot\frac{1}{2}\sqrt{u^2+4} + 3\ln\left|\frac{1}{2}\sqrt{u^2+4} + \frac{u}{2}\right| + c$$

$$I = \sqrt{t^2-6t+13} + 3\ln\left|\frac{1}{2}\sqrt{t^2-6t+13} + \frac{t}{2} - \frac{3}{2}\right| + c$$

or

$$= \sqrt{t^2-6t+13} + 3\ln\left|\sqrt{t^2-6t+13} + t - 3\right| + c.$$

6. (14-points) Use substitution $t = \tan(x/2)$, $-\pi < x < \pi$, to evaluate the integral

$$\int \frac{1}{\sin x + \cos x} dx.$$

$t = \tan(x/2) \Rightarrow \cos x = \dfrac{1-t^2}{1+t^2}$, $\sin x = \dfrac{2t}{1+t^2}$, $dx = \dfrac{2}{1+t^2} dt$

$$I = \int \frac{1}{\sin x + \cos x} dx = \int \frac{2}{1+2t-t^2} dt = 2\int \frac{dt}{2-(t-1)^2}$$

After substitution $u = t-1$, $I = 2\int \dfrac{du}{2-u^2}$

Decompose $\dfrac{1}{2-u^2}$ into partial fractions:

$$\frac{1}{2-u^2} = \frac{1}{(\sqrt{2}-u)(\sqrt{2}+u)} = \frac{A}{\sqrt{2}-u} + \frac{B}{\sqrt{2}+u} \Rightarrow A = B = \frac{1}{2\sqrt{2}}$$

$$I = 2 \cdot \frac{1}{2\sqrt{2}} \left[\int \frac{1}{\sqrt{2}-u} du + \int \frac{1}{\sqrt{2}+u} du \right]$$

$$= \frac{1}{\sqrt{2}} \left[-\ln|\sqrt{2}-u| + \ln|\sqrt{2}+u| \right] + c$$

$$= \frac{1}{\sqrt{2}} \ln\left| \frac{\sqrt{2}+u}{\sqrt{2}-u} \right| + c = \frac{1}{\sqrt{2}} \ln\left| \frac{\sqrt{2}-1+t}{\sqrt{2}+1-t} \right| + c$$

$$= \frac{1}{\sqrt{2}} \ln\left| \frac{\sqrt{2}-1+\tan(x/2)}{\sqrt{2}+1-\tan(x/2)} \right| + C$$

7. (12-points) Evaluate, if possible, the improper integral $\int_0^2 \frac{dx}{\sqrt[3]{|x-1|}}$.

Check the improper integrals $\int_0^1 (x-1)^{-1/3} dx$ and $\int_1^2 (x-1)^{-1/3} dx$.

$$\bullet \lim_{t \to 1^-} \int_0^t (x-1)^{-1/3} dx = \lim_{t \to 1^-} \left[\frac{3}{2} (x-1)^{2/3} \Big|_0^t \right] = \lim_{t \to 1^-} \left(\frac{3}{2} (t-1)^{2/3} - \frac{3}{2} \right) = -\frac{3}{2}$$

Then $\int_0^1 (x-1)^{-1/3} dx = \frac{-3}{2}$

$$\bullet \lim_{t \to 1^+} \int_t^2 (x-1)^{-1/3} dx = \lim_{t \to 1^+} \left[\frac{3}{2} (x-1)^{2/3} \Big|_t^2 \right] = \lim_{t \to 1^+} \left(\frac{3}{2} - \frac{3}{2} (t-1)^{2/3} \right) = \frac{3}{2}$$

Then $\int_1^2 (x-1)^{-1/3} dx = \frac{3}{2}$

Since both improper integrals converge, we can write

$$\int_0^2 \frac{dx}{\sqrt[3]{|x-1|}} = -\int_0^1 \frac{dx}{(x-1)^{1/3}} + \int_1^2 \frac{dx}{(x-1)^{1/3}}$$

$$= -\left(-\frac{3}{2}\right) + \left(\frac{3}{2}\right)$$

$$= 3$$

8. (10-points) Find the value of a so that the length of the arc on the curve $y = \cosh x$ from the point $P(0,1)$ to the point $Q(\ln a, \dfrac{a+a^{-1}}{2})$ is 2.

$$L = \int_0^{\ln a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2$$

$$\frac{dy}{dx} = \sinh x$$

$$1 + \left(\frac{dy}{dx}\right)^2 = \cosh^2 x$$

$$\Rightarrow \int_0^{\ln a} \cosh x \, dx = 2$$

$$\Rightarrow \sinh x \Big|_0^{\ln a} = 2$$

$$\Rightarrow \sinh(\ln a) - \sinh(0) = 2 \Rightarrow \sinh(\ln a) = 2$$

$$\Rightarrow \frac{a - a^{-1}}{2} = 2 \Rightarrow a^2 - 4a - 1 = 0 \Rightarrow a = 2 + \sqrt{5}$$

9. (10-points) Find the area of the surface obtained by revolving the curve $y = 4\sqrt{x+1}$, $0 \le x \le 4$ about the x-axis.

$$S = \int_0^4 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{x+1}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{x+5}{x+1}$$

$$= \int_0^4 2\pi \cdot 4 \cdot \sqrt{x+1} \sqrt{\frac{x+5}{x+1}}\, dx$$

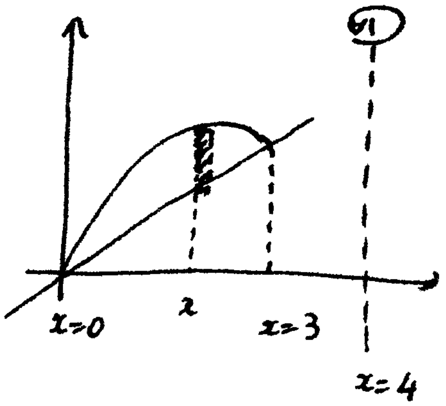
$$= 8\pi \int_0^4 \sqrt{x+5}\, dx = 8\pi \cdot \frac{2}{3} (x+5)^{3/2} \Big|_0^4$$

$$= \frac{16}{3} \pi \left(27 - 5^{3/2}\right)$$

10. (10-points) Use the **method of cylindrical shells** to find the volume generated by rotating the region bounded by the line $y = x$ and the curve $y = 4x - x^2$ about the line $x = 4$.

$4x - x^2 = x \Rightarrow x^2 - 3x = 0 \Rightarrow x = 0$ or $x = 3$

$r = 4 - x \qquad h = (4x - x^2) - x$



$$\text{Volume} = \int_0^3 2\pi (4-x)(3x - x^2)\,dx$$

$$= 2\pi \int_0^3 12x - 7x^2 + x^3\,dx$$

$$= 2\pi \left(6x^2 - \frac{7}{3}x^3 + \frac{1}{4}x^4\right)\Big|_0^3$$

$$= 2\pi \left(54 - 63 + \frac{81}{4}\right)$$

$$= \frac{45\pi}{2}$$