King Fahd University Of Petroleum & Minerals
Department Of Mathematics And Statistics
STAT501 : Probability and Mathematical Statistics (151)
Final Exam **Tuesday December 22, 2015**
Name: **ID:**

Question Number	Full Mark	Marks Obtained
One	17	
Two	16	
Three	18	
Four	18	
Five	15	
Six	17	
Seven	14	
Eight	20	
Nine	16	
Ten	11	
Eleven	18	
Total	180	

Question.1 (**7+7+3=17-Points**)

Suppose that X and Y are two discrete random variables with joint mass function given by the following table:

	Y			
X	1	3	5	6
1	1/9	1/27	1/27	1/9
2	2/9	0	1/9	1/9
3	0	0	1/9	4/27

(a) Find the PMFs of X and Y.

(b) Find $E\{X^2|y=6\}$

(c) Find $P\{X \geq Y\}$

Question.2 (**10+6=16-Points**)

Let (X, Y) be a continuous RV with PDF given by:

$$f(x,y) = \begin{cases} \frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} x^{\alpha_1-1} y^{\alpha_2-1} (1-x-y)^{\alpha_3-1}, & x \geq 0, y \geq 0, x+y \leq 1 \\ 0, & o.w \end{cases}$$

(a) Find the marginal PDF of X

(b) If $\alpha_1 = \alpha_2 = \alpha_3 = 1$, find $E\{X^k\}$ for some positive integer k.

Question.3 (**6+12=18-Points**)

Let $X_1, X_2, \ldots, X_n$ be independent RVs with $X_i \sim Poisson(\lambda_i)$, for $i = 1, 2, \ldots, n$.

(a) If $S = \sum_{i=1}^{n} X_i$, find the PMF of S. (i.e $P\{S = s\}$).

(b) Show that the conditional PMF of $X_1 | S = s$ is $binomial\left(s, \frac{\lambda_1}{\sum_{i=1}^{n} \lambda_i}\right)$

Question.4 (**4+6+5+3=18-Points**)

Let X be a uniform RV on the interval $(0,1)$.

(a) Find the PDF of $Y = -\ln(1-X)$

(b) Let $Y_1, Y_2, \ldots, Y_n$ be *iid* such that $Y_i, i = 1, 2, \ldots, Y_n$ have the same distribution as Y in (a). Let $S = 2\sum_{i=1}^{n} Y_i$. Find the PDF of S

(c) Find $E\{S^k\}$ for the RV in part (b), where k is a positive integer.

(d) Use the result is (c) to find the mean and variance of the RV S

Question.5 (**7+8=15-Points**)

Let X and Y be independent RVs with $X \sim Gamma(r, 1)$ and $Y \sim Gamma(s, 1)$

(a) Let $Z_1 = X + Y$, $Z_2 = \frac{X}{X+Y}$, find the joint PDF of (Z_1, Z_2)

(b) Find the conditional PDF of $Z_1 | Z_2 = z_2$. Are Z_1 and Z_2 independent? Explain.

Question .6 **(5+12=17-Points)**

Let $P_i \sim beta(\alpha, \beta)$, and $X_i|P_i \sim Binomial(1, p_i)$, where $i = 1, 2, \ldots, n$ and $X_i s$ are independent. Define $Y = \sum_{i=1}^{n} X_i$

(a) Find $E\{Y\}$

(b) Find $Var(Y)$

Question .7 **(9+5=14-Points)**

Show that each of the following belongs to the expediential family

(a) $X \sim beta(\alpha, \beta)$

(b) $P\{\mathbf{X} = \mathbf{x}\} = \frac{n!}{x_1!x_2!\ldots x_n!}\alpha_1^{x_1}\alpha_2^{x_2}\ldots\alpha_n^{x_n}$, where $\sum_{i=1}^{n} X_i = n$, and $x_i = 0, 1, \ldots, n$.

Question .8 **(10+10=20-Points)**

Let $X_1, X_2, \ldots$ be *iid* $Uniform(0, \theta)$.Let $X_{(1)} = \min(X_1, X_2, \ldots, X_n)$, and $X_{(n)} = \max(X_1, X_2, \ldots, X_n)$

(a) Let $Y_n = nX_{(1)}$, $n = 1, 2, \ldots$. Show that Y_n converges in distribution to some RV Y and find the DF of Y

(b) Let $Z_n = Y_{(n)}$, $n = 1, 2, \ldots$. Show that $Z_n \xrightarrow{P} \theta$.

Question .9 **(8+8=16-Points)**

(a) Let $\{X_n\}$ be a sequence of pairwise uncorrelated RVs with $E(X_i) = \mu_i$, $Var(X_i) = \sigma_i^2$, $i = 1, 2, \ldots$.

If $\sum_{i=1}^{n} \sigma_i^2 \longrightarrow \infty$ as $n \longrightarrow \infty$. Show that $\dfrac{\sum_{i=1}^{n} (X_i - \mu_i)}{\sum_{i=1}^{n} \sigma_i^2} \xrightarrow{P} 0$

(b) Let $X_1, X_2, \ldots$ be an *iid* sequence of RVs with $E(X_i) = \mu, Var(X_i) = \sigma^2 < \infty$. Let $Y_n = \frac{2}{n(n+2)} \sum_{i=1}^{n} iX_i$. Show that $Y_n \xrightarrow{P} \mu$

Question .10 **(7+4=11-Points)**

(a) Let $X_1, X_2, \ldots$ be an *iid* sequence of RVs with $Var(X_i) = \sigma^2 < \infty$.

Show that $S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X}) \xrightarrow{a.s} var(X_1) = \sigma^2$

(b) If $\{X_n\}$ be a sequence of RVs such that $X_n \xrightarrow{D} N(\mu, \sigma^2)$. Find the limiting distribution of $Y_n = X_n^3 + 3X_n - 1$

Question .11 **(18-Points)**

Let $X_1, X_2, \ldots$ be a sequence of *iid* RVs with common mean μ and variance σ^2. Let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, $S_n^2 = \dfrac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$. Show that $\frac{\sqrt{n}(\overline{X}-\mu)}{S} \xrightarrow{D} Z \sim (0,1)$