King Fahd University of Petroleum & Minerals
Department of Math. & Stat.

Math 668 - Final Exam (151)   Time: 3 hours

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Name: ID #
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Problem 1	/10
Problem 2	/10
Problem 3	/8
Problem 4	/7
Total	/35

Problem # 1. (10 marks) Let Ω be a bounded and smooth domain of $\mathbb{R}^n$. Consider the problem

$$(P_1) \qquad \begin{cases} u_t(x,t) - \left(1 + \int_\Omega |\nabla u(x,t)|^2 dx\right) \Delta u(x,t) = f(x,t) & \text{in } \Omega \times (0,+\infty) \\ u(x,0) = u_0(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times [0,+\infty) \end{cases}$$

where $u_0 \in H_0^1(\Omega)$ and $f \in L^2\left(\Omega \times (0,+\infty)\right)$. Show that (P_1) has a unique weak solution $u \in L^2\left((0,+\infty); H_0^1(\Omega)\right)$ and $u \in L^2\left((0,+\infty); H^{-1}(\Omega)\right)$.

Problem # 2. (10 marks) In a bounded and smooth domain of $\mathbb{R}^n$, consider the problem

$$(\mathrm{P}_2) \qquad \begin{cases} u_{tt}(x,t) + \Delta^2 u(x,t) = f(x,t) & \text{in } \Omega \times (0,+\infty) \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = u = 0 & \text{on } \partial\Omega \times [0,+\infty) \end{cases}$$

If $u_0 \in H^4(\Omega) \cap H_0^2(\Omega)$, $u_1 \in H_0^2(\Omega)$, and $f \in C^1\left([0,+\infty), L^2(\Omega)\right)$, show that (P_2) has a unique solution

$$u \in C\left([0,+\infty), H^4(\Omega) \cap H_0^2(\Omega)\right) \cap C^1\left([0,+\infty), H_0^2(\Omega)\right) \cap C^2\left([0,+\infty), L^2(\Omega)\right)$$

Hint: Use the result if Ω is smooth enough and $v \in H^2(\Omega)$ and $g \in L^2(\Omega)$ satisfying

$$\int_\Omega \Delta v \Delta w + \int_\Omega vw = \int_\Omega gw, \forall w \in H_0^2(\Omega),$$

then $v \in H^4(\Omega)$.

Problem # 3. (8 marks) Given the nonlinear problem

$$(\mathrm{P}_3) \qquad \begin{cases} u_{tt}(x,t) - \Delta u(x,t) + h(u(x,t)) = 0 & \text{in } \Omega \times (0,+\infty) \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times [0,+\infty) \end{cases}$$

where Ω is a bounded and smooth domain of $\mathbb{R}^n$, $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ and $h : \mathbb{R} \to \mathbb{R}$ is continuous such that

$$H(s) \geq 0 \text{ and } |h(s)| \leq C|s|^{p-1},\ |H(s)| \leq C|s|^p,\ C > 0,$$

where $H(s) = \int_0^s h(\xi)d\xi$ and

$$1 < p < \frac{2(n-1)}{n-2},\ \text{if } n \geq 3 \text{ and } p > 1,\ \text{if } n = 1,2$$

Use Galerkin method to establish an existence result for (P_3)

Problem # 4. (7 marks) Let Ω be a bounded and smooth domain of $\mathbb{R}^n$. Consider the problem

$$(P_4) \qquad \begin{cases} u_t(x,t) - div\left(|\nabla u(x,t)|^{m-2}\nabla u(x,t)dx\right) = f(u(x,t)) & \text{in } \Omega \times (0,+\infty) \\ u(x,0) = u_0(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times [0,+\infty) \end{cases}$$

where $f \in C^1(\mathbb{R})$ satisfying, for a constant $p > m > 2$,

$$pF(s) \le sf(s), \ \ F(s) = \int_0 f(\xi)d\xi$$

and $0 \ne u_0 \in W_0^{1,m}(\Omega)$, satisfying

$$\int_\Omega F(u_0(x))dx - \frac{1}{m}\int_\Omega |\nabla u_0(x)|^m dx \ge 0.$$

a. Show that $H'(t) \ge 0$, where

$$H(t) = \int_\Omega F(u(x,t))dx - \frac{1}{m}\int_\Omega |\nabla u(x,t)|^m dx$$

b. Show that $\phi(t) := \frac{1}{2}\int_\Omega u^2(x,t)dx$ satisfies

$$\phi'(t) \ge \left(\frac{p}{m} - 1\right)\left(H(t) + ||\nabla u(x,t)||_m^m\right)$$

c. Show that for some constant $c > 0$,

$$\phi^{\frac{m}{2}}(t) \le c||\nabla u(x,t)||_m^m$$

d. Combine (b) and (c) to show that, for some finite $t^* < +\infty$,

$$\lim_{t \to t^*} \phi(t) = +\infty$$