Problem # 1. (10 marks) Given the problem

$$(P_1)\quad \begin{cases} u_{tt} - u_{xx} + \beta\theta_x = 0, & \text{in } (0,1)\times \mathbb{R}_+\\ \theta_t - b\theta_{xx} + \beta u_{xt} = 0, & \text{in } (0,1)\times \mathbb{R}_+\\ u(0,t) = \theta(0,t) = u_x(1,t) = \theta(1,t) = 0, & \text{on } [0,+\infty)\\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x), & \text{in } (0,1) \end{cases}$$

$b,\ \beta > 0.$

1) Show that the space

$$W = \{w \in H^1(0,1))\ /\ w(0) = 0\}$$

is complete

2) Discuss the well posedness of (P1).

Sol. 1) let (u_n) be a Cauchy sequence in W. so, it is Cauchy in $H^1(I)$. So $u_n \longrightarrow u$ in $H^1(I)$ and $L^\infty(I)$

with $\quad \sup_{x\in I} |u_n(x) - u(x)| \le C\|u_n - u_n\|_{H^1(I)} \longrightarrow 0$

Thus $\quad u_n(0) \longrightarrow u(0) \Rightarrow u(0) = 0 \Rightarrow u \in W$

Hence W is (closed) complete.

2) let $u_t - v = 0$, then $v_t - u_{xx} + \beta\theta_x = 0$

(P_1) takes the form

$$(P_1')\quad \begin{cases} u_t - v = 0\\ v_t - u_{xx} + \beta\theta_x = 0\\ \theta_t - b\theta_{xx} + \beta v_x = 0 \end{cases}$$

let $H = W \times L^2(I) \times L^2(I)$

We rewrite (P_1') as $U_t + AU = 0$,

where $\quad AU = A\begin{pmatrix} u\\ v\\ \theta\end{pmatrix} = \begin{pmatrix} -v\\ -u_{xx} + \beta\theta_x\\ -b\theta_{xx} + \beta v_x\end{pmatrix}$

$$D(A) = \left\{ \begin{pmatrix} u\\ v\\ \theta\end{pmatrix} \in H \Big/ \begin{array}{l} u \in H^2(I),\ v \in W,\ \theta \in H^2(I)\cap H_0^1(I)\\ u_x(1) = 0 \end{array} \right\}$$

We equip H by

$$\left(\begin{pmatrix} u\\ v\\ \theta\end{pmatrix}, \begin{pmatrix} \tilde u\\ \tilde v\\ \tilde\theta\end{pmatrix}\right) = \int_0^1 u_x \tilde u_x + \int_0^1 v\tilde v + \int_0^1 \theta\tilde\theta.$$

Monotonicity: For $V \in D(A)$, we have

$$(AV, V)_H = \left(\begin{pmatrix} -v \\ -u_{xx} + \beta\theta_x \\ -b\theta_{xx} + \beta v_x \end{pmatrix}, \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \right)$$

$$= -\int_0^1 v_x u_x - \int_0^1 u_{xx} v + \beta \int_0^1 \theta_x v - b\int_0^1 \theta\theta_{xx} + \beta \int_0^1 v_x \theta$$

$$= -\cancel{\int_0^1 v_x u_x} - \cancel{u_x v \Big]_0^1} + \cancel{\int_0^1 u_x v_x} - \beta \int_0^1 \theta v_x + b\int_0^1 \theta_x^2 + \beta\int_0^1 \theta v_x$$

$$= b\int_0^1 \theta_x^2 \geq 0 \implies A \text{ is monotone.}$$

Maximality let $F = \begin{pmatrix} f \\ g \\ h \end{pmatrix} \in H$ and find $V \in D(A)$

s.t $V + AV = F$. That is

$$\begin{cases} u - v = f \in W & (1) \\ v - u_{xx} + \beta\theta_x = g \in L^2 & (2) \\ \theta - b\theta_{xx} + \beta v_x = h \in L^2 & (3) \end{cases}$$

(1) + (2) gives $u - u_{xx} + \beta\theta_x = f + g$

We define the bilinear form and linear form as follows:

$$B\left(\begin{pmatrix} u \\ \theta \end{pmatrix}, \begin{pmatrix} \hat{u} \\ \hat{\theta} \end{pmatrix}\right) = \int_0^1 u\hat{u} + \int_0^1 u_x \hat{u}_x + \beta\int_0^1 \theta_x \hat{u} + \int_0^1 \theta\hat{\theta} + b\int \theta_x \hat{\theta}_x$$

$$+ \beta\int_0^1 u_x \hat{\theta}$$

and $$L\begin{pmatrix} \hat{u} \\ \hat{\theta} \end{pmatrix} = \int_0^1 (f+g)\hat{u} + \int_0^1 (h + \beta f_x)\hat{\theta}$$

clearly L and B are linear and bdd on $W \times H_0^1(I)$.

Coerciviness

$$B\left(\begin{pmatrix}u\\ \theta\end{pmatrix},\begin{pmatrix}u\\ \theta\end{pmatrix}\right)=\int_0^1\left(u^2+u_x^2+\theta^2+\theta_x^2\right)\geq \alpha\left\|\begin{pmatrix}u\\ \theta\end{pmatrix}\right\|^2_{W\times H_0^1}$$

Lax - Milgram $\Rightarrow$ $\exists$ ce of a uni que $\begin{pmatrix}u\\ \theta\end{pmatrix}\in W\times H_0^1$ st

$$B\left(\begin{pmatrix}u\\ \theta\end{pmatrix},\begin{pmatrix}\check{u}\\ \check{\theta}\end{pmatrix}\right)=L\left(\begin{pmatrix}\check{u}\\ \check{\theta}\end{pmatrix}\right),\quad \forall\begin{pmatrix}\check{u}\\ \tilde{\theta}\end{pmatrix}\in W\times H_0^1(I). \quad ④$$

Take $\hat{\theta}\equiv 0$ then, we have, by ④

$$\int_0^1 u\hat{u}+\int_0^1 u_x\check{u}_x+\beta\int_0^1\theta_x\hat{u}=\int_0^1(f+g)\check{u},\quad\forall\check{u}\in W \quad ⑤$$

Note that $H_0^1(I)\subset W$. So ⑤ is valid for $\check{u}\in H_0^1(I)$

Thus, we can have

$$\int_0^1 u_x\check{u}_x=\int_0^1(f+g-u-\beta\theta_x)\check{u},\ \forall\ \check{u}\in H_0^1(I)\ (\text{or } C_0^\infty(I))$$

$\Rightarrow$ $u_x\in H^1(I)$ or $u\in H^2(I)$, with

$$u_{xx}=-(f+g-u-\beta\theta_x)$$

or

$$-u_{xx}+u+\beta\theta_x=f+g$$

letting $v=u-f$, we obtain ②.

Next, we integrate by parts in ⑤, to arrive at

$$\int_0^1 u\hat{u}+u_x\tilde{u}\Big|_0^1-\int_0^1 u_{xx}\hat{u}+\beta\int_0^1\theta_x\hat{u}=\int_0^1(f+g)\check{u}$$

That is

$$\int_0^1\underbrace{\left(-u_{xx}+u+\beta\theta_x-f-g\right)}_{0}\check{u}+u_x(1)\check{u}(1)=0,\ \forall\ \check{u}\in W$$

Arbitrareness of $\check{u}$ $\Rightarrow$ $u_x(1)=0$

Thus $u\in W\cap H_0^1(I)$, with $u_x(1)=0$

$$v=u-f\in W$$

Similarly, take $\check{u}\equiv 0$, to get

$$\int_0^1\left(\theta\check{\theta}+\theta_x\check{\theta}_x+\beta v_x\check{\theta}\right)=\int_0^1(h+\beta f_x)\check{\theta},\ \forall\ \check{\theta}\in H_0^1(I)$$

This gives that $\theta \in H_0^2(\Omega)$ and, similarly, integration by parts leads to

$$\int_0^1 (\theta - \theta_{xx} + \beta v_x)\tilde{\theta} = \int_0^1 (h + \beta f_x)\tilde{\theta}, \quad \forall\ \tilde{\theta} \in H_0^1(I)$$

Thus $\quad \theta - \theta_{xx} + \beta u_x = h + \beta f_x$

recalling that $v = u - f$ then we obtain

$$\theta - \theta_{xx} + \beta v_x = h$$

So $U = \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \in D(A)$; hence, A is maximal.

<u>Theorem</u>. suppose that $(u_0, u_1, \theta) \in \mathcal{D}(A)$ then (P_1) has a unique solution s.t

$$u \in C([0,+\infty), H^2 \cap W) \cap C^1([0,+\infty), W) \cap C^2([0,+\infty), L^2(\Omega))$$

$$\theta \in C([0,+\infty), H^2 \cap H_0^1) \cap C^1([0,+\infty), L^2(\Omega)).$$