King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 550: Linear Algebra

Second Exam, Fall Semester 151 (180 minutes)

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Part I. Solve *any* two of Q1, Q2 and Q3:

Q1. (10 points) Let V be an inner product space and $W \leq_F V$ be a subspcace (F is any field). Show that:
(a) If W is finite-dimensional, then $V = W \oplus W^{\perp}$.
(b) IF V is finite-dimensional, then $W = W^{\perp\perp}$.

Q2. (10 points) Let V be a finite-dimensional *real* inner product vector space, $T : V \longrightarrow V$ a self-adjoint linear operator, $c_1, \cdots, c_k$ the distinct eigenvalues of T, and $E_j : V \longrightarrow W_j$ the orthogonal projection of V on W_j (the eigenspace corresponding to c_j) for $j = 1, \cdots, k$. Show that

$$V = W_1 \oplus \cdots \oplus W_k \text{ and } T = c_1 E_1 + \cdots + c_k E_k.$$

Q3. (10 points) Let V be a finite-dimensional complex vector space and f a symmetric bilinear form on V. Show that there exists an ordered basis for V in which f is represented by a diagonal matrix.

Part II. Solve Q4 or Q5:

Let $T : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ be a linear operator represented (w.r.t. the standard basis) by the matrix

$$A = \begin{bmatrix} 0 & -1 & -1 \\ -3 & -1 & -2 \\ 7 & 5 & 6 \end{bmatrix}.$$

Q4. (20 points) (a) Find the Jordan canonical form J of A and an invertible matrix P such that $P^{-1}AP = J$.
(b) classify all *rational* matrices M with

$$\text{char}(M) = \min(M) = (x^2+1)(x^2-2)(x-3)^2$$

by specifying their possible rational canonical forms.

Q5. (20 points) (a) Find the rational canonical form B of A and a basis β of $\mathbb{C}^3$ such that $[T]_\beta = B$.
(b) Classify all *real* matrices N with

$$\text{char}(N) = (x-2)^4(x-3)^3 \text{ and } \min(N) = (x-2)^2(x-3)^3$$

by specifying their possible *Jordan canonical forms.*

Part III. Solve *all* of the following questions:

Q6. (20 points) Let

$$V = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

equipped with the inner product

$$(f \mid g) = \int_0^1 f(x)g(x)dx.$$

and $W = \mathbb{R}$ (the subspace of V consisting of all scalar polynomials).

(a) Find $W^{\perp}$ (provide a basis).

(b) Apply the Gram-Schmidt Process to the basis $\{1, x, x^2, x^3\}$ of V.

Q7. (10 points) Let V be a finite dimensional vector space and f a non-degenerate symmetric bilinear form on V. Show that for each linear operator $T \in L(V, V)$, there is a *unique* linear operator $T' \in L(V, V)$ such that

$$f(T(\alpha), \beta) = f(\alpha, T'(\beta)) \text{ for all } \alpha, \beta \in V.$$

Q8. (10 points) Let V be a finite dimensional vector space. Show that

$$g : L(V, V) \longrightarrow L(V^*, V^*),\ T \longmapsto T^t$$

is a linear isomorphism.

Q9. (20 points) Prove or disprove:

(1) There are $n \times n$ complex matrices A and B such that $AB - BA = I$.

(2) Every linear operator on an inner product space has an adjoint.

(3) If V is a finite dimensional inner product space, then

$$V^* = \{(-, \beta) \mid \beta \in V\}.$$

(4) There exists a 5×5 *real* matrix A with $\min(A) = x^2 + x + 1$.

GOOD LUCK