**King Fahd University of Petroleum & Minerals**

**Department of Mathematics and Statistics**

**Math 550: Linear Algebra**

**Second Exam, Fall Semester 151 (120 minutes)**

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**Q1. (20 points)** Let $V$ be a vector space.

(a) Show that if $W$ is a *hyperspace* in $V$, then there exists $g \in V^* \backslash \{0\}$ such that $W = N_g$ (the nullspace of $g$).

(b) Let $V$ and $W$ be finite dimensional vector spaces and $T : V \longrightarrow W$ a linear transformation. Show that

$$\text{rank}(T^t) = \text{rank}(T).$$

**Q2. (20 points)** Let $V$ be a finite dimensional vector space and $T$ a diagonalizable linear operator on $V$.

(a) Show that the minimal polynomial of $T$ is a product of distinct linear functions.

(b) Let $c_1, \cdots, c_k$ be the distinct eigenvalues of $T$ and $W_i$ be the eigenspace corresponding to $c_i$ for $i = 1, \cdots, k$. Show that there exists a set of *orthogonal projections* $E_1, \cdots, E_k$ such that:

(i) $E = c_1E_1 + \cdots + c_kE_k$

(ii) $I = E_1 + \cdots + E_k$

(iii) $\text{Range}(E_i) = W_i$.

**Q3. (20 points)** Consider the linear operator

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ (x, y, z) \mapsto (6x + z, 3x - 2y, -8x - 3z).$$

(a) Find the eigenvalues for $T$ and the bases for the corresponding eigenspaces.

(b) Show that $T$ is *not* diagonalizable over $\mathbb{R}$. Justify your answer.

(c) Show that $T$ is triangulable over $\mathbb{R}$. Justify your answer.

**Q4. (15 points)** Consider the linear operator

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ (x, y, z) \mapsto (3x + 2z, y, -x).$$

(a) Show that $T$ is diagonalizable over $\mathbb{R}$.

(b) Find a matrix $P$ such that $P^{-1}AP$ is diagonal, where $A$ is the matrix representing $T$ w.r.t. the standard basis.

**Q5. (10 points)** Let $V$ be a finite dimensional vector space and $T$ a linear operator on $V$ with $\text{rank}(T) = 1$. Show that $T$ is either nilpotent or diagonalizable.

**Q6. (15 points)** True of False? Justify your answer correcting the wrong one(s).

(a) Every linear operator on a vector space over $\mathbb{C}$ is triangulable.

(b) For every vector space $V$, we have $V \simeq V^{**}$.

(c) If $T$ is a linear operator on the finite dimensional vector space $V = W_1 \oplus W_2$, where the subspaces $W_1$, $W_2$ are invariant under $T$, then

$$\min(T) = \min(T_{|W_1})\ \min(T_{|W_2}).$$

**GOOD LUCK**