King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
MATH533 - Complex Variables
Final Exam – Semester I, 2015-2016

Exercise 1

Let $a > 0$. Evaluate

$$\int_0^{+\infty} \frac{\cos ax}{(1+x^2)^2}\, dx$$

Exercise 2

Determine the number of roots, counted with multiplicity, of the equation

$$2z^5 - 6z^2 + z + 1 = 0$$

in the annulus $A(0,1,2) = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

Exercise 3

(a) Find and classify all isolated singularities of

$$f(z) = \frac{z^2(z-\pi)}{\sin^2 z} \quad \text{and} \quad g(z) = (z^2-1)\cos\frac{1}{z-1}$$

(b) Find the residue of f at $z = 2\pi$ and the residue of g at $z = 1$.

Exercise 4

If $P(z) = a_0 + a_1 z + \ldots + a_n z^n$, evaluate

$$S_P := \frac{1}{2\pi i} \int_{|z|=R} z \, \frac{P'(z)}{P(z)} dz$$

for large values of R. Deduce the value of S_P, for $P(z) = z^n - 1$, $n \geq 2$.

Exercise 5

(a) Consider a rational function $f(z) = q(z)/p(z)$, where p is a polynomial of degree n and q is a polynomial of degree $n-2$ or less. If $z_1, z_2, \ldots, z_n$ are distinct roots of p, prove that the residues of f satisfy $\sum_{k=1}^{n} \operatorname{Res}(f, z_k) = 0$.

(b) Show that there is a holomorphic function in the set $\Omega = \{z \in \mathbb{C} : |z| > 4\}$ whose derivative is $\dfrac{z}{(z-1)(z-2)(z-3)}$.

Is there a holomorphic function on Ω whose derivative $\dfrac{z^2}{(z-1)(z-2)(z-3)}$.

Exercise 6

(a) If $w(z)$ is an entire function and $p(z)$ is a polynomial such that $|w(z)| \leq |p(z)|$ for all $z \in \mathbb{C}$. Show that $w(z) = cp(z)$, where $c \in \mathbb{C}$, with $|c| \leq 1$.

(b) What we can say about w, if the estimate $|w(z)| \leq |p(z)|$ holds for $|z|$ large ?

Exercise 7

(a) Prove that if f is an entire function without any zeros then there is an entire function g such that $f(z) = e^{g(z)}$.

(b) What can you say about f If, $f(0) = 0$ and $f(z) \neq 0$ for $z \neq 0$.

Exercise 8

(a) Let f be an analytic function in the disk $\{|z| < R\}$ satisfying $|f(z)| < M$. Suppose that $f(z_0) = 0$ for some z_0, $|z_0| < R$. Show that then

$$|f(z)| \leq \frac{MR|z - z_0|}{|R^2 - z\bar{z}_0|}$$

for any z, $|z| < R$, and

$$|f'(z_0)| \leq \frac{MR}{R^2 - |z_0|^2}.$$

(b) Denote by Δ the unit disk, $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Does there exist an analytic function $f : \Delta \to \Delta$ with $f(1/2) = 3/4$ and $f'(1/2) = 2/3$?