King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics
MATH 302
Final Exam
2015-2016 (151)

Thursday, December 24, 2015 **Allowed Time: 3 Hours**

Name: ____

ID Number: ______________________ Serial Number: _______________

Section Number: _____________ Instructor's Name: _________________

Instructions:

1. Write neatly and legibly. You may lose points for messy work.

2. **Show all your work.** No points for answers without justification.

3. **Programmable Calculators and Mobiles are not allowed**.

4. Make sure that you have 10 different problems (10 pages + cover page).

Problem No.	Points	Maximum Points
1		15
2		8
3		22
4		10
5		11
6		9
7		10
8		18
9		17
10		20
Total:		140

Q1. Let $\mathbf{A} = \begin{pmatrix} 2 & -2 & 1 \\ -1 & 3 & -1 \\ 2 & -4 & 3 \end{pmatrix}$. Find a matrix **P** that diagonalizes **A** and the diagonal matrix **D** such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$

Solution:

The characteristic equation of A is

$$|A - \lambda I| = -\lambda^3 + 8\,\lambda^2 - 13\,\lambda + 6 = -(\lambda - 6)(\lambda - 1)^2 = 0.$$

This gives the eigenvalues of A: $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 6$.

The corresponding eigenvectors are

$$K_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},\ K_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\ \ and\ \ K_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Thus, a matrix P that diagonalizes A is given by

$$P = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$$

and the diagonal matrix D is

$$D = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

Q2. Let C be the line segment from $(1,2,0)$ to $(-2,-1,0)$. Evaluate $\int_C (x-y)\,dy$.

Solution:

Parametric equations for the line segment are

$$x = 1 + (-2-1)t = 1 - 3t,$$

$$y = 2 + (-1-2)t = 2 - 3t,$$

$$z = 0 + (0-0)t = 0, \;\; 0 \le t \le 1.$$

We obtain

$$\int_C (x-y)\,dy = \int_0^1 3\,dt = 3$$

Q3. Let D be the region bounded by the paraboloid $z = x^2 + y^2 + 1$ and the plane $z - 2x = 4$. Use the **divergence theorem** to find the outward flux $\iint_S (\mathbf{F} \cdot \mathbf{n})\, dS$ of the vector field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} - \mathbf{k}$, where S is the boundary of D.

Solution:

In Cartesian coordinates the region D is given by

$$D = \left\{ \begin{matrix} (x, y, z) | -1 \le x \le 3, -\sqrt{3 - x^2 + 2x} \le y \le \sqrt{3 - x^2 + 2x}, \\ 1 + x^2 + y^2 \le z \le 4 + 2x \end{matrix} \right\}.$$

The divergence of F is

$$\text{div } F = 2.$$

By the divergence theorem,

$$\iint_S (\mathbf{F} \cdot \mathbf{n})\, dS = \iiint_D \text{div } F\, dV = 2 \iiint_D dV$$

$$= 2 \int_{-1}^{3} \int_{-\sqrt{3-x^2+2x}}^{\sqrt{3-x^2+2x}} (3 + 2x - x^2 - y^2) dy dx$$

$$= 2 \int_{-1}^{3} \left[(3 + 2x - x^2) y - \frac{1}{3} y^3 \right]_{-\sqrt{3-x^2+2x}}^{\sqrt{3-x^2+2x}} dx$$

$$= \frac{8}{3} \int_{-1}^{3} (3 - x^2 + 2x)^{\frac{3}{2}}\, dx$$

$$= \frac{8}{3} \int_{-1}^{3} [4 - (x - 1)^2]^{\frac{3}{2}}\, dx \quad (\text{let } s = x - 1)$$

$$= \frac{8}{3} \int_{-2}^{2} [4 - s^2]^{\frac{3}{2}}\, ds \qquad (\text{let } s = 2 \sin \theta)$$

$$= \frac{8}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 16 \cos^4\theta\, d\theta = \frac{8}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4(1 + 2\cos 2\theta + \cos^2 2\theta)\, d\theta$$

$$= \frac{32}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2} \cos 4\theta \right) d\theta = \frac{64}{3} \left[\frac{3}{2} \theta + \sin 2\theta + \frac{1}{8} \sin 4\theta \right]_0^{\frac{\pi}{2}} = 16\pi.$$

Another solution:

In Cartesian coordinates the region D is given by

$$D = \{(x, y, z) | (x-1)^2 + y^2 = 4, 1 + x^2 + y^2 \leq z \leq 4 + 2x\}.$$

The divergence of F is

$$\text{div } F = 2.$$

By the divergence theorem,

$$\iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS \quad = \iiint_D \text{div } F \, dV = 2 \iiint_D \, dV$$

$= 2 \iint_R (3 + 2x - x^2 - y^2) dA$ where $R = \{(x, y) | (x-1)^2 + y^2 = 4\}$.

let $s = x - 1$, then the integral

$= 2 \iint_M (4 - s^2 - y^2) dA$ where $M = \{(s, y) | s^2 + y^2 = 4\}$.

$$= 2 \int_0^{2\pi} \int_0^2 (4 - r^2) r dr d\theta$$

$$= 4\pi \int_0^2 (4r - r^3) dr$$

$$= 4\pi \left[2r^2 - \frac{r^4}{4}\right]_0^2$$

$= 16\pi$.

Q4. Let $f(z) = |z|^2 + i$. Show that $f(z)$ is differentiable only at $z_0 = 0$ and find $f'(0)$.

Solution:

Let $z = x + i\,y$. We have

$$f(z) = |z|^2 + i = x^2 + y^2 + i.$$

Put $u = x^2 + y^2$ and $v = 1$. We obtain

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial x} = 0,$$

Since the Cauchy-Riemann equations hold only at point $(0,0)$, then $f(z)$ is not differentiable at any $z \neq 0$.

Moreover, since $u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous in a neighborhood about a point $(0,0)$, then $f(z)$ is differentiable at $z = 0$ and $f'(0) = \frac{\partial u}{\partial x}(0,0) + i\frac{\partial v}{\partial x}(0,0) = 0$.

Q5. Find all complex numbers z satisfying $\cos z = 2$.

Solution:

$$\frac{e^{iz}+e^{-iz}}{2}=2$$

$$e^{iz}+e^{-iz}-4=0$$

$$\left(e^{iz}\right)^2-4e^{iz}+1=0$$

$$e^{iz}=\frac{4\pm\sqrt{16-4}}{2}=2\pm\sqrt{3}$$

$$iz=ln\left(2\pm\sqrt{3}\right)$$

$$iz=log_e\left|2\pm\sqrt{3}\right|+i(0+2n\pi),\qquad n=0,\pm1,\pm2,\ldots$$

$$z=2n\pi-i\ log_e\left(2\pm\sqrt{3}\right),\qquad n=0,\pm1,\pm2,\ldots$$

Q6. Find $\int_C \frac{\sin|z|}{z}\,dz$, where C is the circular arc (in the first quadrant) along $|z| = 2$ from $z = 2$ to $z = 2\,i$.

Solution:

On curve C, we have $z = 2(\cos t + \; i \sin t) = 2e^{it}, 0 \leq t \leq \pi/2$ and $dz = 2ie^{it}dt$.

Thus,

$$\int_C \frac{\sin|z|}{z}\,dz$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin 2}{2e^{it}}\, 2ie^{it}dt$$

$$= i \sin 2 \int_0^{\frac{\pi}{2}} dt$$

$$= i\,\frac{\pi}{2} \sin 2.$$

Q7. Let $f(z) = z^5 \sin\left(\frac{1}{z^2}\right)$.

(a) Expand $f(z)$ as Laurent series valid for $0 < |z|$.

Solution:

We want all powers of z in the series. We obtain

$$f(z) = z^5 \sin\left(\frac{1}{z^2}\right)$$

$$= z^5 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{1}{z^2}\right)^{2k+1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{3-4k}$$

$$= z^3 - \frac{1}{3!}\frac{1}{z} + \frac{1}{5!}\frac{1}{z^5} - \cdots$$

The series valid for the annular domain $0 < |z|$.

(b) Evaluate $\oint_C f(z)\,dz$ where C is a positively oriented rectangle containing zero.

Solution:

Since the residue at $z = 0$ of the given function is $a_{-1} = -\frac{1}{6}$, then

$$\oint_C f(z)\,dz = 2\pi i \operatorname{Res}\left(f(z), 0\right) = -\frac{1}{3}\pi i.$$

Q8. Use the **Cauchy integral formula** (for higher derivatives) to evaluate the integral $\oint_C \left(\frac{z^3-2}{z^3(z-1)} - z^3 \cos z\right) dz$ where C is the positively oriented circle

$|z-1| = 2.$

Solution:

Let $I_1 = \oint_C \frac{z^3-2}{z^3(z-1)}\, dz$ *and* $I_2 = \oint_C z^3 \cos z\, dz$.

Since $z^3 \cos z$ is an entire function, then $I_2 = 0$ by Cauchy theorem.

Since the two singularities $z = 0$ and $z = 1$ are lying within C, then we can use the deformation theorem as follows:

$$I_1 = \oint_{C_1} \frac{z^3-2}{z^3(z-1)}\, dz + \oint_{C_2} \frac{z^3-2}{z^3(z-1)}\, dz$$

where $C_1\colon |z| = 0.1$ and $C_2\colon |z-1| = 0.1$.

Moreover, one can rewrite the integral I_1 as follows:

$$I_1 = \oint_{C_1} \frac{f_1(z)}{z^3}\, dz + \oint_{C_2} \frac{f_2(z)}{z-1}\, dz$$

Where $f_1(z) = \frac{z^3-2}{z-1}$ is analytic on C_1 and $f_2(z) = \frac{z^3-2}{z^3}$ is analytic on C_2.

Thus, we have $f_1^{(2)}(z) = \frac{2z^3-6z^2+6z-4}{(z-1)^3}$. Using the Cauchy's integral formula, we obtain

$$I_1 = \frac{2\pi i}{2!} f_1^{(2)}(0) + 2\pi i f_2(1) = 4\pi i - 2\pi i = 2\pi i$$

Hence,

$$\oint_C \left(\frac{z^3-2}{z^3(z-1)} - z^3 \cos z\right) dz = I_1 + I_2 = 2\pi i + 0 = 2\pi i.$$

Q9. Use the **residue theorem** to evaluate $\oint_C \frac{\sin z}{z^2(z+i)^2}\, dz$ where C is the positively oriented circle $|z| = 3$.

Solution:

Observe that $f(z) = \frac{sin\, z}{z^2(z+i)^2}$ has two singularities $z = 0$ and $z = -i$ and both of them lie within C. Thus,

$$I = \oint_C \frac{sin\, z}{z^2(z+i)^2}\, dz = 2\pi i\, [Res(f(z), 0) + Res(f(z), -i)]$$

Since $z = 0$ is a zero of the denominator of order 2 and zero of the numerator of order 1, then $z = 0$ is a simple pole and

$$Res(f(z), 0) = \lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{sin\, z}{z(z+i)^2} = -1$$

Since $z = -i$ is a zero of the denominator of order 2 and is not a zero of the numerator, then $z = -i$ is a double pole and

$$Res(f(z), 0) = \lim_{z\to -i} \frac{d}{dz}\left((z+i)^2 f(z)\right)$$

$$= \lim_{z\to -i} \frac{d}{dz}\left(\frac{sin\, z}{z^2}\right) = \lim_{z\to -i} \frac{z^2\, cos\ z - 2\, z\, sin\, z}{z^4} = -cos\ \ i - 2\, i\, sin\, i$$

Hence,

$I = -1 - cos\ \ i - 2\, i\, sin\, i.$

Q10. Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{1}{x^4+16} dx$.

Solution:

Let $f(z) = \frac{1}{z^4+16}$.

Since $z^4 + 16 = (z - z_0)(z - z_1)(z - z_2)(z - z_3)$ where $z_k = 2e^{\frac{\pi i + 2k\pi i}{4}}, k = 0,1,2,3.$

We let C be the closed contour consisting of the interval $[-R, R]$ on the x-axis and the semi-circle C_R of radius $R > 2$ given by $z = Re^{it}, t = 0 \dots \pi$.

Then

$$\oint_C f(z)\, dz = \int_{-R}^{R} \frac{1}{x^4+16} dx + \int_{C_R} \frac{1}{z^4+16}\, dz,$$

Since degree $z^4 + 16$ minus degree $1 \geq 2$, then $\lim_{R\to\infty} \int_{C_R} \frac{1}{z^4+16}\, dz = 0$ and so

$$P.V. \int_{-\infty}^{\infty} \frac{1}{x^4+16} dx = 2\pi i\, [Res(f(z), z_0) + Res(f(z), z_1)].$$

Since $z = z_0$ and $z = z_1$ are simple poles, then

$$Res(f(z), z_0) = \frac{1}{4{z_0}^3} = \frac{1}{32} e^{-\frac{3\pi i}{4}} = \frac{1}{32}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$Res(f(z), z_1) = \frac{1}{4{z_1}^3} = \frac{1}{32} e^{-\frac{9\pi i}{4}} = \frac{1}{32} e^{-\frac{\pi i}{4}} = \frac{1}{32}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

Hence,

$$P.V. \int_{-\infty}^{\infty} \frac{1}{x^4+16} dx = 2\sqrt{2}\pi\,.$$