King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics
MATH 302
EXAM II
2015-2016 (151)

Wednesday, November 11, 2015 **Allowed Time: 2 Hours**

Name: Key

ID Number: ______________ **Serial Number:** ______________

Section Number: ______________ **Instructor's Name:** ______________

Instructions:

1. Write neatly and legibly. You may lose points for messy work.

2. **Show all your work.** No points for answers without justification.

3. **Programmable Calculators and Mobiles are not allowed.**

4. Make sure that you have 7 different problems (5 pages + cover page).

Problem No.	Points	Maximum Points
1		9
2		9
3		9
4		15
5		16
6		21
7		21
Total:		100

Q1. Let $G(x,y,z) = x^2 - 2y + e^{yz}$. Find the direction, in which the function $G(x,y,z)$ increases most rapidly at the point (1,1,0). What is the maximum rate of change in this case?

$\nabla G(x,y,z) = \langle 2x, -2 + z e^{yz}, y e^{yz} \rangle$

The required direction is $\nabla G(1,1,0) = \langle 2,-2,1 \rangle$

and the maximum rate is $|\nabla G(1,1,0)| = \sqrt{4+4+1} = 3$

Q2. Find the length of the curve given by $r(t) = (\frac{1}{2}t^2 - t)\,\mathbf{i} - \frac{2}{\sqrt{3}}t^{\frac{3}{2}}\,\mathbf{j} + \frac{2}{3}t^{\frac{3}{2}}\,\mathbf{k}$, $0 \le t \le 1$.

length $= \int_C |r'(t)|\,dt = \int_0^1 |r'(t)|\,dt$

$r'(t) = \langle t-1, \sqrt{3}\,t^{\frac{1}{2}}, t^{\frac{1}{2}} \rangle = (t-1)\vec{i} + \sqrt{3}\,t^{\frac{1}{2}}\vec{j} + t^{\frac{1}{2}}\vec{k}$

$|r'(t)| = \sqrt{(t-1)^2 + 3t + t} = \sqrt{t^2 + 2t + 1} = t+1.$

$\therefore$ length $= \int_0^1 (t+1)\,dt = \frac{1}{2}(t+1)^2 \Big|_0^1 = \frac{1}{2}(4-1) = 3/2.$

Q3. Find the equations of the normal line and the tangent planes to the surface

$z = x\,y$ at the point (1,1,1).

The surface is given by $F(x,y,z) = z - xy = 0$

$\nabla F(x,y,z) = \langle -y, -x, 1 \rangle$

$\nabla F(1,1,1) = \langle -1, -1, 1 \rangle$

Equation of tangent plane

$-(x-1) - (y-1) + (z-1) = 0 \iff z - x - y + 1 = 0$

Equations of the normal line

$x = -t + 1$

$y = -t + 1$

$z = t + 1$

Q4. On the unit sphere, the gravitational attraction force between a mass m_1 at the origin and a mass m_2 at the sphere is given by the vector field $\mathbf{F} = -\boldsymbol{c}(y\,z\,\mathbf{i} + x\,z\,\mathbf{j} + x\,y\,\mathbf{k})$ where $\boldsymbol{c}$ is a positive constant and (x, y, z) is the coordinate of m_2.

i) Show that F is conservative.

$$\text{Curl } F = -c\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = -c\langle x-x,\ y-y,\ z-z\rangle = \vec{0}.$$

P, Q, R and their derivatives are continuous. So F is conservative

ii) Find a potential $\emptyset(x, y, z)$ whose gradient is the vector field **F**.

$$\nabla\phi = F \iff \phi_x = -cyz,\ \phi_y = -cxz,\ \phi_z = -cxy$$

$$\phi_x = -cyz \implies \phi = -cxyz + \alpha(y,z)$$

$$\implies \phi_y = -cxz + \alpha_y = -cxz \implies \alpha_y(y,z) = 0$$

$$\implies \alpha(y,z) = \beta(z) \implies \phi = -cxyz + \beta(z)$$

$$\implies \phi_z = -cxy + \beta'(z) = -cxy \implies \beta'(z) = 0 \implies \beta(z) = c^t$$

So a potential could be

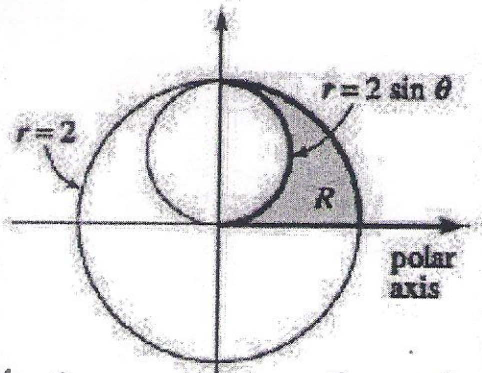
$$\boxed{\phi(x,y,z) = -cxyz}$$

iii) Evaluate the work done by the force $\boldsymbol{F}$ in moving a mass m_2 from the point $A(\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}})$ to the point $B(0, -1, 0)$.

$$\text{Work} = \int_A^B F\cdot dr = \phi(B) - \phi(A)$$

$$= 0 + \frac{c}{4\sqrt{2}} = \frac{c}{4\sqrt{2}}.$$

Q5. Use **Green's Theorem** to evaluate $\oint_C (2y + \sin(x^3))dx + (7x + \cos(y^5))\, dy$, where C consists of the boundary of the shaded region $\boldsymbol{R}$ with a counterclockwise orientation.



Notice that R is a simply connected region enclosed by a simple closed positively oriented path C.

So, we can use Green's theorem since $P, Q, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ are continuous.

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_R (7-2)\, dA = 5 \iint_R dA$$

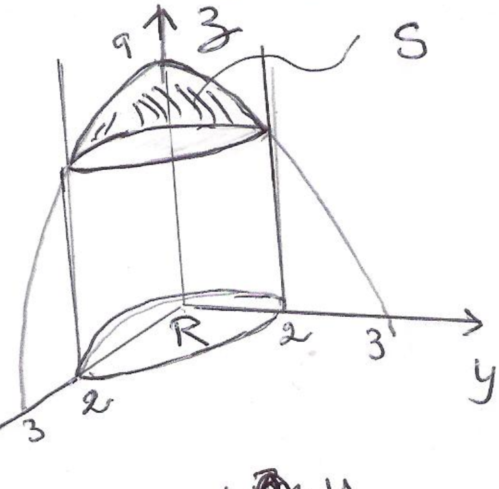
$$= 5 \int_0^{\pi/2} \int_{2\sin\theta}^{2} r\,dr\,d\theta = 5 \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_{2\sin\theta}^{2} d\theta$$

$$= 5 \int_0^{\pi/2} (2 - 2\sin^2\theta)\, d\theta = 10 \int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= 10 \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta = 5 \int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta$$

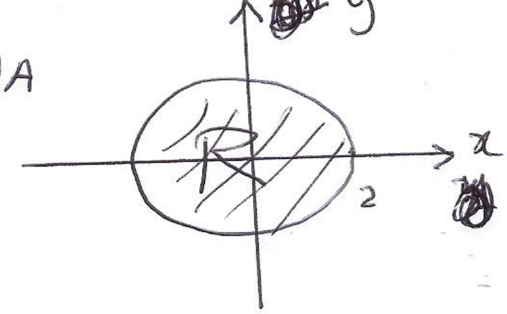
$$= 5 \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{5\pi}{2}$$

Q6. Find $\iint_S (9-z)\,dS$ where S is the portion of the paraboloid $z = 9 - x^2 - y^2$ inside the cylinder $x^2 + y^2 = 4$.



The projection of S is the disk R in the xy plane.

S is given by $z = f(x,y)$
$= 9 - x^2 - y^2$



So

$$I = \iint_S (9-z)\,ds = \iint_R (9-z)\sqrt{1+f_x^2+f_y^2}\;dA$$

on $S:- \; z = 9 - x^2 - y^2$

$$\sqrt{1+f_x^2+f_y^2} = \sqrt{1+4x^2+4y^2}$$

Thus $$I = \iint_R (x^2+y^2)\sqrt{1+4(x^2+y^2)}\;dA$$

$$= \int_0^{2\pi}\int_0^2 r^2\sqrt{1+4r^2}\,r\,dr\,d\theta \qquad \text{(polar coordinates)}$$

let $u = 1+4r^2 \Rightarrow r^2 = \dfrac{u-1}{4}$ and $du = 8r\,dr$

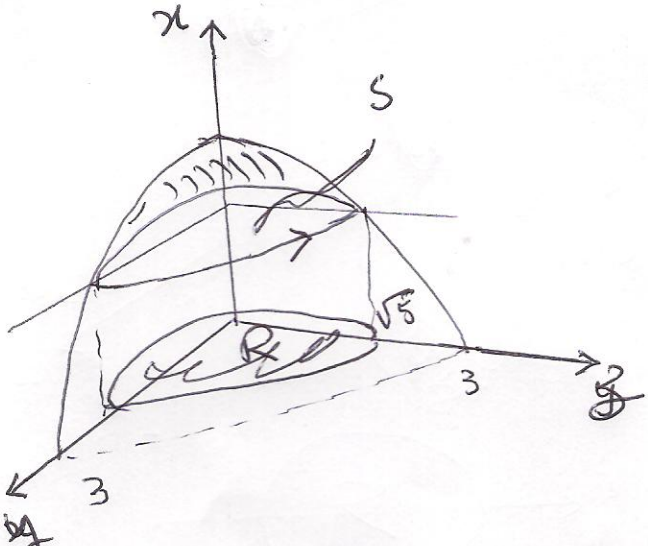
Therefore, $$I = \int_0^{2\pi}\int_1^{17} \frac{u-1}{4}\cdot u^{\frac{1}{2}}\,\frac{du}{8}\,d\theta$$

$$= 2\pi\cdot\frac{1}{32}\int_1^{17}\left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du = \frac{\pi}{16}$$

$$= \frac{\pi}{16}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^{17} = \frac{\pi}{60}\left(391\sqrt{17}+1\right)$$

Q7. Use **Stokes' theorem** to evaluate $\oint_C z^2 x\,dx + z e^{x^2}\,dy + \tan^{-1} x\,dz$ where C is the curve of intersection of the plane $x = 2$ and the sphere $x^2 + y^2 + z^2 = 9$, by finding a surface S with C as its boundary and such that the orientation of C is counterclockwise as viewed from above.

$$W = \oint_C F \cdot dr = \iint_S \text{Curl } F \cdot \eta \, dS$$



$$\text{Curl } F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 x & z e^{x^2} & \tan^{-1} x \end{vmatrix}$$

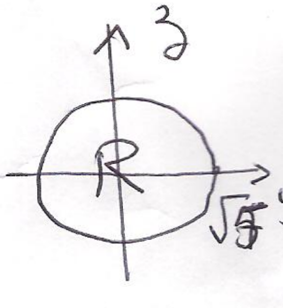
$$= \langle -e^{x^2},\ 2zx - \frac{1}{1+x^2},\ 2xze^{x^2} \rangle$$

let's take S to be the disk on the plane $x = 2$

When $x = 2$ then $x^2 + y^2 + z^2 = 9 \Rightarrow y^2 + z^2 = 5$

Thus the projection of S on the yz-plane is the disk

$$R = \{(y,z) \mid z^2 + y^2 \le 5\}.$$



S is given by $x = f(y,z) = 2$

Thus

$$W = \iint_S \text{curl } F \cdot \eta \, dS = \iint_S e^x \, dS, \quad \eta = \langle 0, 0, 1 \rangle$$

$$= \iint_R e^{x^2} \sqrt{1 + f_y^2 + f_z^2}\, dA = \iint_R e^4 \, dA$$

$$= e^4 \text{ area}(R) = e^4 \pi (\sqrt{5})^2 = 5\pi e^4.$$